On the Limiting Behaviors and Positivity of Quasi-local Mass

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Abstract

The concept of quasi-local masses was proposed by physicists about forty years ago to measure the energy of a given compact region by a closed spacelike 2-surface. There are several natural conditions which we expect a quasi-local mass to satisfy ([39]):

1. A quasi-local mass must be non-negative in general and zero when, and only when the ambient spacetime of the surface is the Minkowski spacetime in the asymptotically flat case (or hyperbolic space in the asymptotically hyperbolic case). These are called the positivity and rigidity conditions.

2. Also, the ADM mass should be recovered as the surfaces tend to the spacial infinity.

In this thesis, we will report some results about the limiting behaviors and positivity of some quasi-local masses, both in the asymptotically flat case and in the asymptotically hyperbolic case.

摘要

大概四十年前，物理學家引入準局域量的概念，去量度一個被閉二維類空表面包圍的空間能量。一般來說，我們期望準局域量有如下性質([39])：

1. 準局域量一般是非負的，並且在漸近平坦的情況下，當且僅當該表面的環繞空間是Minkowski時空它才是零(或在漸近雙曲的情況下是雙曲空間)。

2. 當曲面趨向無窮時，準局域量應趨向全域的質量(ADM mass)。

在這篇論文裡，我們會講述在漸近平坦或漸近雙曲的情況下，關於準局域量的一些極限特性和非負特性的結果。

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Contents

Introduction

As is well known, by the equivalence principle in general relativity, the concept of gravitational energy at a point is not well defined. The local effects of gravity can be removed by using a freely falling frame. The object centered at the origin of such a frame will not experience any gravitational acceleration.

On the other hand, when there is asymptotic symmetry (asymptotically flat or hyperbolic), the concepts of total energy and momentum can be well defined. In the asymptotically flat case, these are the so called ADM energy momentum [2] and the Bondi energy-momentum when the system is viewed from spatial infinity and null infinity, respectively. It was proved by Bartnik [3] that in an asymptotically flat manifold, these concepts are well-defined, i.e. independent of the coordinates chosen. These concepts are fundamental in general relativity and have been proven to be natural. Moreover, the works of Schoen-Yau [31, 32], Witten [41] show that they satisfy the important positivity condition. These kinds of results are now known as positive mass theorems. However, when the physical system is not isolated, or the asymptotic symmetry fails, there would be limitations to these concepts. It was proposed about forty years ago to measure the energy of a system by enclosing a region with a "membrane", i.e. a closed spacelike 2-surface, and define on it an energy-momentum 4-vector. This is the motivation behind the definition of quasi-local masses of surfaces.

There are several natural conditions which we expect a quasi-local mass to satisfy (see for example [39]):

1. Most importantly, a quasi-local mass must be non-negative in general and zero when, and only when the ambient spacetime of the surface is the Minkowski spacetime (or hyperbolic space in the asymptotically hyperbolic case). These are called the positivity and rigidity conditions.

2. Also, the ADM mass or Bondi mass should be recovered as the surfaces

tend to the spacial or null infinity.

There is still no universal agreement on the definition of the quasi-local mass, and many other definitions have been proposed, for example from Hawking [13] and Penrose [27]. A promising one was given by Brown-York [7], motivated by Hamiltonian formulation. Shi and Tam [34] proved that it is positive in the time symmetric case, but in general it is not positive. Later on, Wang and Yau [38] proposed the notion of Wang-Yau mass and proved its positivity and rigidity. The study of these quasi-local masses and their relations is now a subject under intense study.

In this thesis, we will establish some results about the limiting behaviors and positivity of some quasi-local masses in asymptotically flat (AF) or asymptotically hyperbolic (AH) manifolds.

In Chapter 1, we will discuss the limiting behaviors of the Brown-York quasi-local mass of some family of surfaces. As mentioned before, we expect that the quasi-local mass of the boundary of exhausting domains tends to the ADM mass. Indeed, many people have proved that the Brown-York quasi-local mass of the coordinate sphere tends to the ADM mass in an AF manifold, see for example the works of Brown-York [8], Hawking-Horowitz [14], Baskaran-Lau-Petrov [4], Shi-Tam [34] and also Fan-Shi-Tam [12]. Shi-Wang-Wu [36] also proved that the same result is true even for surfaces which are not necessarily coordinate spheres, but are nearly round near infinity.

The motivation of investigating the Brown-York mass for some general class of surfaces is as follows. In [3], Bartnik proved the following important result (see Theorem 1.3 for a more precise statement):

Theorem 0.1. *Suppose (M,γ) is an AF manifold with integrable scalar curvature. Let $\{D_k\}$ be an exhaustion of M by closed sets such that the set $S_k=\partial D_k$ are connected C^1 surfaces (not necessarily coordinate spheres) satisfying some* *reasonable conditions. Then*

$$m_{ADM}(M,\gamma) = \lim_{k\to\infty} \frac{1}{16\pi} \int_{S_k} \sum_{i,j=1}^{3} (\gamma_{ij,i} - \gamma_{ii,j})\, \nu^j d\sigma^0.$$

That is, the ADM mass is independent of the sequence of $\{S_k\}$*. (Note that ADM mass is defined exactly as the R.H.S. of the above equation, except that* S_k *are coordinate spheres.)*

Because of this result, it is natural to ask if the Brown-York quasi-local mass of some general family of surfaces, other than those which are close to the coordinate spheres, will tend to the ADM mass in some AF manifolds. We will see in this chapter that this is true for certain kinds of revolution surfaces, for example a family of expanding ellipsoids, which are not close to the coordinate spheres. More precisely we will prove the following

Theorem A. *[Theorem 1.6, Limiting behaviors in AF case] If* (N^3, g) *is an asymptotically Schwarzchild manifold and* S *is a given closed revolution surface* S*. Then there is an* $\varepsilon > 0$ *such that for any family of revolution surfaces* S_a *with Gaussian curvatures of order* $O(a^{-2})$*, mean curvatures of order* $O(a^{-1})$ *and radial distances of order* $O(a)$*, if the rescaled surfaces* $a^{-1}S_a$ *are* ε*-close to* S*, then the Brown-York masses of the surfaces will tend the ADM mass:*

$$\lim_{a\to\infty} m_{BY}(S_a) = m_{ADM}(N,g).$$

This partly generalizes the results of [6, 34, 12].

In Chapter 2, we will work in the asymptotically hyperbolic (AH) case. The motivation of this chapter is quite similar to the previous chapter. In particular, this is partly motivated by the positive mass theorem proved by X.D. Wang [40] in an AH manifold. Let us first recall the positive mass theorem in an AF manifold: if we are given a complete asymptotically flat initial data set (M^3, g, h) for the Einstein equations, we can then define the total 4-momentum (E, P) of (M^3, g, h), where $P \in \mathbb{R}^3$. The positive mass theorem of Schoen-Yau [31, 33, 32] then states that

Theorem 0.2. *Let (M^3, g, h) be an asymptotically flat initial data set satisfying the dominated energy condition (e.g. non-negative scalar curvature when $h = 0$), then $E \geq |P|$.*

This can be interpreted as the 4-momentum being a future directed non-spacelike vector in $\mathbb{R}^{3,1}$. Later on, this result was reproved by Witten [41] (under spin condition) using a different proof involving spinors. The spinor method turns out to be very useful in proving positive mass type theorems. In particular we have the following result of X.D. Wang, which can be regarded as the analogue of Schoen-Yau's result in the AF case:

Theorem 0.3. *[40, Theorem 2.5] If (M^n, g) is spin, asymptotically hyperbolic and the scalar curvature $R \geq -n(n-1)$, then the total mass defined by (see Theorem 2.2 for precise definitions)*

$$(\int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h) d\mu_{g_0}, \int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h) x d\mu_{g_0}) \in \mathbb{R}^{n,1}$$

is a future-directed non-spacelike vector.

In an AH manifold, we can define a quasi-local mass integral which is similar to the Brown-York mass in the AF case. Let (Ω, g) be a three dimensional compact manifold with smooth boundary Σ homeomorphic to sphere. Under certain conditions, Σ can be uniquely embedded into $\mathbb{H}^3 \subset \mathbb{R}^{3,1}$. Then the quasi-local mass integral of Ω is defined as:

$$\int_{\Sigma} (H_0 - H) X \tag{1}$$

where H_0 is the mean curvature of Σ in $\mathbb{H}^3$ and $X = (x^0, x^1, x^2, x^3)$ is the position vector in $\mathbb{R}^{3,1}$.

The motivation of this definition is as follows. In [35], Shi and Tam proved that if the scalar curvature of Ω satisfies $R \geq -6$, then the vector $\int_{\Sigma}(H_0 - H)W$ is a future directed non-spacelike vector for $W(x^0, x^1, x^2, x^3) = (\alpha x^0, x^1, x^2, x^3)$ with $\alpha > 1$ depending on the geometry of Σ. (This is exactly Theorem C when

$n = 3$.) Hence W is close to the position vector. It is also conjectured by Shi and Tam that the same result is true if W is replaced by the position vector X. It is therefore natural to ask if the quasi-local mass integral defined as in (1) for coordinate spheres will tend to the total mass as defined in Theorem 0.3. We will give a positive answer to this question. Namely, we will show that

Theorem B. *[Theorem 2.3, Limiting behaviors in AH case] In an asymptotically hyperbolic manifold (M^3, g), for a coordinate sphere S_r which is close enough to infinity, we can associate with it a quasi-local mass expression (as a vector in $\mathbb{R}^{3,1}$), which will tend to the total mass of (M^3, g) defined by Theorem 0.3 when S_r approaches infinity.*

Whereas the first two chapters deals with the limiting behaviors of the quasi-local masses, in Chapter 3 we will look at the positivity of a quasi-local mass. This chapter is also closely related to Chapter 2. As mentioned before, Witten [41] (see also [26, 3]) gave a simplified proof of the positive mass theorem using the spinor method. Since then the method of spinor has been adopted by many people to prove positive mass type theorems or some rigidity results [34, 1, 23, 38]. For example, M. T. Wang and Yau [38] developed a quasi-local mass for a three dimensional manifold with boundary whose scalar curvature is bounded from below by some negative constant. Using spinor method, they were able to prove that this mass is non-negative. Later on, Shi and Tam [35] also proved a similar result in the three dimensional case, but with a simpler definition of the mass. More precisely, they proved the following:

Theorem 0.4. *([35] Theorem 3.1) Let (Ω, g) be a compact orientable 3-dimensional manifold with smooth boundary $\Sigma = \partial\Omega$, homeomorphic to a 2-sphere. Assuming the following conditions:*

1. *The scalar curvature R of (Ω, g) satisfies $R > -6k^2$ for some $k > 0$,*

2. *Σ is a topological sphere with Gaussian curvature $K > -k^2$ and mean*

curvature $H > 0$, so that Σ can be isometrically embedded into $\mathbb{H}^3_{-k^2}$ with mean curvature H_0.

Then there is a future directed time-like vector-valued function W on Σ such that the vector

$$\int_\Sigma (H_0 - H) W d\Sigma$$

is time-like. Here $W = (x_1, x_2, x_3, \alpha t)$ for some $\alpha > 1$ depending only on the intrinsic geometry of Σ, with $X = (x_1, x_2, x_3, t)$ being the future-directed unit normal vector of $\mathbb{H}^3_{-k^2}$ (defined in (3.1)) in $\mathbb{R}^{3,1}$.

In this chapter, we will prove an analogous result in higher dimension for spin manifolds (note that three-dimensional orientable manifolds are spin) as follows.

Theorem C. *[Theorem 3.16, Positivity of Shi-Tam mass] Let $n \geq 3$ and (Ω, g) be a compact spin n-manifold with smooth boundary Σ such that*

1. *The scalar curvature R of (Ω, g) satisfies $R > -n(n-1)k^2$ for some $k > 0$,*
2. *Σ is topologically a $(n-1)$-sphere with sectional curvature $K > -k^2$, mean curvature $H > 0$ and Σ can be isometrically embedded uniquely into $\mathbb{H}^n_{-k^2} \subset \mathbb{R}^{n,1}$ with mean curvature H_0.*

Under these conditions, we can define on Σ a quasi-local mass introduced by Shi and Tam [35]:

$$m_{ST}(\Sigma) = \int_\Sigma (H_0 - H) W \in \mathbb{R}^{n,1}$$

where $W = (x_1, x_2, \cdots, x_n, \alpha t)$ with $\alpha > 1$ depending on the geometry of Σ and $(x_1, x_2, \cdots, x_n, t)$ is the position vector of Σ in $\mathbb{R}^{n,1}$.

Then the mass is positive in the sense that $m_{ST}(\Sigma)$ is a future directed non-spacelike vector in $\mathbb{R}^{n,1}$.

There are two important ingredients in establishing the main result. One is a monotonicity formula (Lemma 3.6) for the mass expression, and the other is a positive mass type theorem (Theorem 3.7). The later is particularly important. This theorem was originally proved by M.T. Wang and Yau [38] in the three dimensional case. Here we will give a proof in general dimension. In particular, the Killing spinor fields play an important role in the proof, and we will give a detailed study on them. What is new in the proof of the theorem in higher dimension are two identities involving Killing spinors on the hyperbolic space (Proposition 3.9 and 3.10).

Theorem A, Theorem B and Theorem C, which are the main results of this thesis, will be proved in Chapter 1, 2 and 3 respectively.

Chapter 1

Brown-York mass in AF manifolds

1.1 Asymptotically flat manifolds

In this chapter, we will discuss the limiting behaviors of the quasi-local mass of a family of surfaces in an asymptotically flat manifold. Let us first recall some definitions. We will adopt the following definition of asymptotically flat manifolds.

Definition 1.1. *A complete three dimensional manifold (M, γ) is said to be asymptotically flat (AF) of order τ (with one end) if there is a compact subset K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^3 \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates in $\mathbb{R}^3$, the metric γ satisfies:*

$$\gamma_{ij} = \delta_{ij} + \sigma_{ij} \tag{1.1}$$

with

$$|\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial\partial\sigma_{ij}| + r^3|\partial\partial\partial\sigma_{ij}| = O(r^{-\tau}), \tag{1.2}$$

for some constant $\frac{1}{2} < \tau \leq 1$.

Here r and ∂ as the Euclidean distance and the standard derivative operator on $\mathbb{R}^3$ respectively, δ is the usual Euclidean metric.

A coordinate system of M near infinity so that the metric tensor in this system satisfy the above decay conditions is said to be admissible. In such a coordinate system, we can define the ADM mass as follows.

Definition 1.2. *The Arnowitt-Deser-Misner (ADM) mass (see [2]) of an asymptotically flat manifold (M, γ) is defined as:*

$$m_{ADM}(M,\gamma) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} (\gamma_{ij,i} - \gamma_{ii,j})\, \nu^j d\sigma^0, \tag{1.3}$$

where S_r is the Euclidean sphere, $d\sigma^0$ is the area element of S_r induced by the Euclidean metric, ν is the outward unit normal of S_r in $\mathbb{R}^3$ and the derivative is the ordinary partial derivative.

To see that this gives a reasonable definition of mass, let us look at the Schwarzschild metric. On a three dimensional slice of Schwarzschild spacetime, corresponding to time = constant, the metric is given by $(1 + \frac{m}{2r})^4\delta$ (using the convention that $G = c = 1$), where m is the mass of a star (as $r \to \infty$, its limit becomes the Newtonian model of a point mass with mass m). It is easily calculated that the integral on the R.H.S. of (1.3) also tends to m as $r \to \infty$. Thus the ADM mass gives a reasonable definition of mass, at least in this case.

We always assume that the scalar curvature is in $L^1(M)$ so that the limit exists in the definition. In [3], Bartnik showed that the ADM mass is a geometric invariant. More precisely, he proved the following theorem (see [3, Proposition 4.1] for a more general setting):

Theorem 1.3. *Suppose (M, γ) is an AF manifold with scalar curvature $R(\gamma) \in L^1(M)$. Let $\{D_k\}_{k=1}^{\infty}$ be an exhaustion of M by closed sets such that the set $S_k = \partial D_k$ are connected C^1 surfaces without boundary in $\mathbb{R}^3$ such that*

$$r_k = \inf\{|x|, x \in S_k\} \to \infty \text{ as } k \to \infty$$
$$r_k^{-2} Area(S_k) \text{ is bounded as } k \to \infty.$$

Then

$$m_{ADM}(M,\gamma) = \lim_{k\to\infty} \frac{1}{16\pi} \int_{S_k} (\gamma_{ij,i} - \gamma_{ii,j})\, \nu^j d\sigma^0.$$

That is, the ADM mass is independent of the sequence of $\{S_k\}$.

Next, let us recall the definition of the Brown-York quasi-local mass. Suppose (Ω, γ) is a compact three dimensional manifold with smooth boundary $\partial\Omega$, if moreover $\partial\Omega$ has positive Gauss curvature, then the Brown-York mass of $\partial\Omega$ is defined as (see [7, 8]):

Definition 1.4.

$$m_{BY}(\partial\Omega) = \frac{1}{8\pi}\int_{\partial\Omega}(H_0 - H)d\sigma \tag{1.4}$$

where H *is the mean curvature of* $\partial\Omega$ *with respect to the outward unit normal and the metric* γ, $d\sigma$ *is the area element induced on* $\partial\Omega$ *by* γ *and* H_0 *is the mean curvature of* $\partial\Omega$ *when embedded in* $\mathbb{R}^3$.

The existence of an isometric embedding in $\mathbb{R}^3$ (Weyl's embedding theorem) for $\partial\Omega$ was proved by Nirenberg [25], the uniqueness of the embedding was given by [15, 30, 29], so the Brown-York mass is well-defined.

It can be proved that the Brown-York mass and the Hawking quasi-local mass [13] of the coordinate spheres tends to the ADM mass in some AF manifolds, see [8, 14, 6, 4, 34, 12], and even of nearly round surfaces [36]. It is therefore natural to ask whether the quasi-local mass of a more general class of surfaces tends to the ADM mass.

In the coming sections, we will consider a special class of AF manifolds, called asymptotically Schwarzschild manifolds, which is defined as follows:

Definition 1.5. $(N, \widetilde{g})$ *is called an asymptotically Schwarzschild manifold if* $N = \mathbb{R}^3 \setminus K$, K *is a compact set containing the origin, and*

$$\widetilde{g}_{ij} = \phi^4\delta_{ij} + b_{ij}, \phi = 1 + \frac{m}{2r}, m > 0,$$

where $|b_{ij}| + r|\partial b_{ij}| + r^2|\partial\partial b_{ij}| + r^3|\partial\partial\partial b_{ij}| = O\left(r^{-2}\right)$.

Clearly, $(N, \widetilde{g})$ is an AF manifold. For $b = 0$, $(N, \widetilde{g})$ is called a Schwarzschild manifold. In this case, we always denote $\widetilde{g}$ as g. Note that the scalar curvature of (N, g) is zero [19] (page 283) and that of $(N, \widetilde{g})$ is in $L^1(N)$, so in both cases the ADM mass is well defined.

1.2 Brown-York mass of revolution surfaces

In this section, we will study the limiting behaviors of Brown-York mass on some family of convex revolution surfaces in an asymptotically Schwarzschild manifold. Our main result is the following:

Theorem 1.6. *[11] Let $(N, \widetilde{g})$ be an asymptotically Schwarzschild manifold and S be a $C^{6,\alpha}$ $(0 < \alpha < 1)$ closed convex revolution surface parametrized by*

$$(\overline{w}(\varphi)\cos\theta, \overline{w}(\varphi)\sin\theta, \overline{h}(\varphi)), \quad 0 \le \theta \le 2\pi \text{ and } 0 \le \varphi \le l. \tag{1.5}$$

Then there exists $\varepsilon > 0$ such that for any family of $C^{5,\alpha}$ closed convex revolution surfaces S_a in $(\mathbb{R}^3, \delta)$ satisfying the following conditions:
(i)

$$\overline{K} \ge \frac{C_1}{a^2} \tag{1.6}$$

where $\overline{K}$ is the Gaussian curvature of S_a with induced Euclidean metric.
(ii)

$$0 < \overline{H} \le \frac{C_2}{a} \tag{1.7}$$

where $\overline{H}$ is the mean curvature of S_a with induced Euclidean metric.
(iii)

$$C_3 a \le \min_{x\in S_a} r(x) \le \max_{x\in S_a} r(x) \le C_4 a, \tag{1.8}$$

where $C_i > 0$ are independent of a for $i = 1, 2, 3, 4$.
Suppose also that (by applying a rotation if necessary) S_a is parametrized by

$$(aw_a(\varphi)\cos\theta,\ aw_a(\varphi)\sin\theta,\ ah_a(\varphi)),\ 0 \le \theta \le 2\pi \text{ and } 0 \le \varphi \le l$$

such that

$$|w_a - \overline{w}|_{C^4} + |h_a - \overline{h}|_{C^4} < \varepsilon \quad \text{for sufficiently large } a. \tag{1.9}$$

Then

$$\lim_{a\to\infty} m_{BY}(S_a) = m_{ADM}(N, \widetilde{g}).$$

From this result, one has

Corollary 1.7. *Let $(N, \widetilde{g})$ be an asymptotically Schwarzschild manifold. Let $\{S_i\}$ be a family of C^7 closed convex revolution surfaces in $(\mathbb{R}^3, \delta)$ satisfying (1.6)-(1.8) and is parametrized as:*

$$(a_i w_i(\varphi)\cos\theta,\ a_i w_i(\varphi)\sin\theta,\ a_i h_i(\varphi))\,,\ 0 \le \theta \le 2\pi \text{ and } 0 \le \varphi \le l$$

for some constant $l > 0$, here a_i are positive numbers with $\lim\limits_{i\to\infty} a_i = +\infty$. If there is a constant c such that

$$|w_i|_{C^7} + |h_i|_{C^7} \le c$$

for all i, then there is a subsequence $\{S_{i_k}\}$ of $\{S_i\}$ such that

$$\lim_{k\to\infty} m_{BY}\left(S_{i_k}\right) = m_{ADM}(N, \widetilde{g}).$$

To prove Theorem 1.6, we will show that we can actually reduce the case to which the ambient space is Schwarzschild. The main proposition is the following:

Proposition 1.8. *Let (N, g) be a Schwarzschild manifold. Suppose $\{S_a\}_{a>0}$ is a family of closed convex surfaces of revolution in $(\mathbb{R}^3, \delta_{ij})$ with the rotation axis passing through the origin, satisfying (1.6)-(1.8). Then*

$$\lim_{a\to\infty} m_{BY}(S_a) = m_{ADM}(N, g).$$

Remark 1.9. *The conditions (i) and (ii) in Theorem 1.6 imply that the principal curvature λ of S_a in $(\mathbb{R}^3, \delta)$ satisfy $\frac{C_1}{C_2 a} \le \lambda \le \frac{C_2}{a}$ for any a. For, if $0 < \lambda_1 \le \lambda_2$ are the principal curvatures, then (1.7) implies $\lambda_2 \le \frac{C_2}{a}$. Together with (1.6), $\lambda_1 \ge \frac{C_1}{\lambda_2 a^2} \ge \frac{C_1}{C_2 a}$.*

Remark 1.10. *By condition (i) of Theorem 1.6 and the Gauss-Bonnet theorem, the Euclidean area of S_a is of order $O(a^2)$.*

We will first show in Subsection 1.2.1 how our main result follows from Proposition 1.8 by a perturbation argument. We will then prove our Proposition 1.8 in Subsection 1.2.3. To do this, we need some estimates for the embeddings and the various curvatures of S_a, which will be done in Subsection 1.2.2.

One example of surfaces satisfying the conditions in Theorem 1.6 is the family of ellipsoids:

$$S_a = \left\{ (x^1)^2 + (x^2)^2 + \frac{(x^3)^2}{4} = a^2 \right\},$$

which is not nearly round [36]. In contrast, the Hawking mass of this family of ellipsoids in (N, g) does not tend to the ADM mass of (N, g), indeed one can check that the Hawking mass [13] of this family tends to negative infinity as $a \to \infty$.

1.2.1 Reduction to the Schwarzschild case

In this subsection, we will reduce the case of Theorem 1.6 to the Schwarzschild case. Let us first compare the mean curvatures of S_a under different metrics.

Lemma 1.11. *For the surfaces S_a satisfying the conditions in Theorem 1.6, we have*

$$|\widetilde{H} - H| \leq Ca^{-3}$$

for some constant C independent of a, where $\widetilde{H}$ and H are the mean curvatures of S_a with respect to $\widetilde{g}$ and g respectively.

Proof. We claim that

$$|\widetilde{A} - A|_g = O\left(a^{-3}\right) \tag{1.10}$$

where A and $\widetilde{A}$ are the second fundamental forms with respect to g and $\widetilde{g}$ respectively.

Let $\rho(x)$ defined on N to be the distance from x to S_a with respect to $\widetilde{g}$. We will use the fact [18, (7.10)]:

$$\widetilde{A}(X, Y) - |\nabla \rho|_g A(X, Y) = \left(\Gamma^k_{ij} - \widetilde{\Gamma}^k_{ij}\right) X^i Y^j \rho_k \tag{1.11}$$

for any tangent vectors X, Y of S_a. For completeness, we prove it here. We proceed as in [36] Lemma 2.6. First of all, we have

$$
\begin{aligned}
A(X,Y) = g\left(\nabla_X\left(\frac{\nabla\rho}{|\nabla\rho|_g}\right), Y\right) &= \frac{g(\nabla_X(\nabla\rho), Y)}{|\nabla\rho|_g} \\
&= \frac{X(Y(\rho)) - (\nabla_X Y)(\rho)}{|\nabla\rho|_g} \\
&= \frac{X^iY^j\rho_{ij} - X^iY^j\Gamma_{ij}^k\rho_k}{|\nabla\rho|_g},
\end{aligned} \tag{1.12}
$$

here the subscript denotes ordinary derivative and Γ_{ij}^k are the Christoffel symbols with respect to g, with the indices $i, j, k = 1, 2, 3$. Denote $\widetilde{\Gamma}_{ij}^k$ to be the Christoffel symbols with respect to $\widetilde{g}$. Then since the $\widetilde{g}$ gradient $|\widetilde{\nabla}\rho|_{\widetilde{g}} = 1$, we also have

$$
\widetilde{A}(X,Y) = X^iY^j\rho_{ij} - X^iY^j\widetilde{\Gamma}_{ij}^k\rho_k.
$$

Combining this with (1.12), we can get (1.11).

Note that $|\Gamma_{ij}^k - \widetilde{\Gamma}_{ij}^k| = O\left(r^{-3}\right)$ by the assumptions of the metrics. By asymptotic flatness, $1 = \widetilde{g}^{ij}\rho_i\rho_j \geq C\sum\rho_i^2$, so $|\rho_i|$ is uniformly bounded. The condition $\widetilde{g}_{ij} = g_{ij} + b_{ij}$ implies $|\widetilde{g}^{ij} - g^{ij}| = O\left(r^{-2}\right)$, so

$$
||\nabla\rho|_g^2 - 1| = |(g^{ij} - \widetilde{g}^{ij})\rho_i\rho_j| = O\left(r^{-2}\right)
$$

which implies

$$
|\nabla\rho|_g = 1 + O\left(r^{-2}\right).
$$

Finally, the principal curvatures $\overline{\lambda}_i$ in Euclidean metric are of order $O\left(a^{-1}\right)$ by Remark 1.9, the principal curvatures λ_i with respect to g are related to $\overline{\lambda}_i$ by ([19] Lemma 1.4): $\lambda_i = \phi^{-2}\overline{\lambda}_i + 2\phi^{-3}n(\phi)$ where n is the unit outward normal with respect to δ. In particular, as $n(\phi) = O(a^{-2})$,

$$
|A|_g = O(a^{-1}).
$$

Combining all these together with (1.11), it is easy to see that (1.10) holds. Combining (1.10) and the metric conditions of g and $\widetilde{g}$ in Definition 1.5, this implies the lemma. $\square$

Let $(S_a, d\widetilde{s}^2)$, (S_a, ds^2) denote the surface S_a with metric $d\widetilde{s}^2, ds^2$ induced from $\widetilde{g}$, g respectively. By Lemma 1.15, for $a >> 1$, the Gaussian curvatures on $(S_a, d\widetilde{s}^2)$ and (S_a, ds^2) are both positive, which implies that they can be isometrically embedded into $(\mathbb{R}^3, \delta)$ uniquely. Now let us compare the mean curvature after embedding:

Lemma 1.12. *Under the same notations and conditions of Theorem 1.6. Let* $\widetilde{H}_0$, H_0 *be the mean curvature of the embedded surfaces of* $(S_a, d\widetilde{s}^2)$ *and* (S_a, ds^2) *in* $(\mathbb{R}^3, \delta)$ *respectively, as* $a >> 1$, *we have* $|\widetilde{H}_0 - H_0| \leq C_5 a^{-3}$ *for some constant* $C_5(S)$.

Proof. We can set $\hat{\varphi} = \frac{\pi}{l}\varphi$, so it suffices to show that the lemma holds for $l = \pi$. Also, by identifying S and S_a with the sphere $\mathbb{S}^2$, we can regard all the metrics here (ds^2 etc.) to be metrics on $\mathbb{S}^2$. We will denote w_a as w and h_a as h. Similar to (1.22), one has

$$\begin{cases} d\overline{s}^2 & = a^2\left((w'^2 + h'^2)\, d\varphi^2 + w^2 d\theta^2\right), \\ d\overline{s}_S^2 & = \left(\overline{w}'^2 + \overline{h}'^2\right) d\varphi^2 + \overline{w}^2 d\theta^2 \end{cases}$$

which are the metrics on S_a and S induced from the Euclidean metric respectively. By definition,

$$ds^2 = \phi^4 d\overline{s}^2,\ d\widetilde{s}^2 = ds^2 + b, \quad \text{where } b = b_{ij}dx^i dx^j\big|_{S_a} \text{ on } S_a.$$

From (1.9), w and its derivatives up to forth order are uniformly bounded for $a >> 1$, the same holds for h. By the conditions of b_{ij}, it is easy to see that the followings hold:

$$\|a^{-2}d\widetilde{s}^2 - a^{-2}ds^2\|_{C^3} = a^{-2}\|b\|_{C^3} \leq C_6 a^{-2}, \tag{1.13}$$

$$\|a^{-2}ds^2 - a^{-2}d\overline{s}^2\|_{C^3} = a^{-2}\|(\phi^4 - 1)d\overline{s}^2\|_{C^3} \leq C_6 a^{-1} \tag{1.14}$$

for some constant $C_6(S)$. By (1.9), we have

$$\|a^{-2}d\overline{s}^2 - d\overline{s}_S^2\|_{C^3} \leq C_7\varepsilon \tag{1.15}$$

for some constant $C_7(S)$. So for $a >> 1$, by (1.14) and (1.15), we have

$$\|a^{-2}ds^2 - d\overline{s}_S^2\|_{C^3} \leq (C_6 + C_7)\,\varepsilon.$$

By the result of [24] Lemma 5.3, if we choose some $0 < \varepsilon < \dfrac{\delta}{\pi^{1-\alpha}(C_6 + C_7)}$ such that

$$\|a^{-2}ds^2 - d\overline{s}_S^2\|_{C^{2,\alpha}} < \delta$$

for sufficiently large a, where δ is the one given by [24] Lemma 5.3, then there are isometric embeddings $\widetilde{X}$ and X of $(\mathbb{S}^2, a^{-2}d\widetilde{s}^2)$ and $(\mathbb{S}^2, a^{-2}ds^2)$ respectively, such that by (1.13), for sufficiently large a,

$$\|\widetilde{X} - X\|_{C^{2,\alpha}} \leq C_8\|a^{-2}d\widetilde{s}^2 - a^{-2}ds^2\|_{C^{2,\alpha}} = O\left(a^{-2}\right)$$

for some constant $C_8(S)$. Since $a\widetilde{X}, aX$ are the isometric embeddings of $(\mathbb{S}^2, d\widetilde{s}^2)$ and $(\mathbb{S}^2, ds^2)$ respectively. Hence $|\widetilde{H}_0 - H_0| = O\left(a^{-3}\right)$. The lemma holds. □

Now we can prove Theorem 1.6.

Proof of Theorem 1.6. By Proposition 1.8, we know that

$$\lim_{a\to\infty} \frac{1}{8\pi}\int_{S_a} (H_0 - H)\, d\sigma = m_{ADM}(N, g).$$

Since the ADM mass of (N, g) is equal to that of $(N, \widetilde{g})$, combining with Lemma 1.11 and Lemma 1.12, we can get the result. □

1.2.2 Estimates for the curvatures and embeddings of S_a

For simplicity, from now on to the end of this chapter, we use $O\left(a^k\right)$ to denote a quantity which is bounded by Ca^k for some constant C independent of a as a is sufficiently large. We will first compute the mean curvature of S_a in (N, g) and of the embedded surface of the Euclidean space respectively.

From the assumptions of S_a, we can assume that S_a is parametrized by

$$(aw_a(\varphi)\cos\theta, aw_a(\varphi)\sin\theta, ah_a(\varphi)), \quad 0 \leq \varphi \leq l_a, 0 \leq \theta \leq 2\pi,$$

$w_a(\varphi), h_a(\varphi)$ being smooth functions for $\varphi \in [0, l_a]$ (i.e. w_a, h_a can be extended smoothly on a slightly larger interval). Moreover,

(i)

$$\begin{aligned} h_a(0) &> h_a(l_a) \\ C_3 \le \sqrt{w_a^2 + h_a^2} &\le C_4 \\ w_a > 0 \text{ on } &(0, l_a), \end{aligned} \tag{1.16}$$

(ii) The generating curve $(w_a(\varphi), h_a(\varphi))$ is parameterized by arc length. i.e.

$$w_a'^2 + h_a'^2 = 1. \tag{1.17}$$

(iii) w_a is anti-symmetric about 0 and l_a, h_a is symmetric about 0 and l_a, i.e.

$$\begin{aligned} &w_a(-\varphi) = -w_a(\varphi), \quad w_a(l_a + \varphi) = -w_a(l_a - \varphi), \\ &h_a(-\varphi) = h_a(\varphi), \quad h_a(l_a + \varphi) = h_a(l_a - \varphi) \text{ for } \varphi \in [0, \varepsilon). \end{aligned} \tag{1.18}$$

This implies

$$w_a(0) = w_a(l_a) = h_a'(0) = h_a'(l_a) = 0. \tag{1.19}$$

Since S_a is convex in $(\mathbb{R}^3, \delta)$ and the Gaussian curvature $\overline{K}$ of S_a with the induced metric $d\overline{s}^2$ is

$$\overline{K} = \frac{h_a'(w_a'h_a'' - w_a''h_a')}{a^2 w_a} \quad \text{for } \varphi \in (0, l_a).$$

So $h_a' < 0$ for $\varphi \in (0, l_a)$ by (1.16).

Let ϕ_a be the function ϕ restricted on S_a, note that in (φ, θ) coordinates, $\phi_a = \phi_a(\varphi)$ is independent of θ. We have the following lemma:

Lemma 1.13. *The functions* $\dfrac{w_a}{h_a'}$ *and* $\dfrac{\phi_a'}{h_a'}$ *can be extended continuously to the whole* $[0, l_a]$*. Moreover there exists a constant* C *independent of* a *such that for all* a*,*

$$\left| \frac{w_a}{h_a'} \right| \le C, \quad \left| \frac{\phi_a'}{h_a'} \right| \le \frac{C}{a}.$$

Proof. We first show that the limits $\displaystyle\lim_{\varphi \to 0} \frac{w_a}{h_a'}$ and $\displaystyle\lim_{\varphi \to l_a} \frac{w_a}{h_a'}$ exist and are uniformly bounded.

The Gaussian curvature $\overline{K}$ of the point $(0,0,ah_a(0))$ on S_a with induced Euclidean metric is equal to $\overline{K}=\frac{h_a''(0)^2}{a^2}$. (This can be seen by noting that for an arc-length parametrized plane curve $(w_a(\varphi),h_a(\varphi))$, its curvature is given by $-w_a''h_a'+h_a''w_a'$.) So at $(0,h_a(0))$, its curvature is $h_a''(0)$.

As $\overline{K}\geq\dfrac{C_1}{a^2}$ by (1.6), $|h_a''(0)|\geq\sqrt{C_1}>0$. By L'Hospital rule,

$$\lim_{\varphi\to 0}\frac{w_a}{h_a'}=\frac{w_a'(0)}{h_a''(0)}$$

which is finite and is bounded by some $C>0$ by (1.6) and (1.17). The same applies to $\lim\limits_{\varphi\to l_a}\dfrac{w_a}{h_a'}$.

Next, observe that one of the principal curvatures of S_a in $(\mathbb{R}^3,\delta)$ is $-\dfrac{h_a'}{aw_a}$ ([10] p.162, (10)). So by Remark 1.9, we have $\left|\frac{w_a}{h_a'}\right|\leq C$ on the whole $[0,l_a]$ for all a.

By differentiating $\phi_a=1+\dfrac{m}{2a\sqrt{w_a^2+h_a^2}}$, $\frac{\phi_a'}{h_a'}=-\frac{m}{2a(w_a^2+h_a^2)^{\frac{3}{2}}}(w_a'\frac{w_a}{h_a'}+h_a)$ which can be extended to $[0,l_a]$ by the above, and is of order $O\left(a^{-1}\right)$ by (1.16), (1.17). $\square$

We have the following estimates

Lemma 1.14. *Regarding $\phi_a=\phi_a(\varphi)$ as functions on S_a, we have $\phi_a'=O(a^{-1})$ and $\phi_a''=O(a^{-1})$.*

Proof. Let $A=w_a^2+h_a^2$. As $\phi_a=1+\dfrac{m}{2a\sqrt{A}}$, we only have to prove $(A^{-\frac{1}{2}})'=O(1)$ and $(A^{-\frac{1}{2}})''=O(1)$. By direct computations and (1.16), (1.17),

$$|(A^{-\frac{1}{2}})'|=|A^{-\frac{3}{2}}(w_aw_a'+h_ah_a')|\leq A^{-\frac{3}{2}}(w_a^2+h_a^2)^{\frac{1}{2}}(w_a'^2+h_a'^2)^{\frac{1}{2}}=O(1).$$

$$\begin{aligned}|(A^{-\frac{1}{2}})''|&=\left|\frac{3}{2}A^{-\frac{5}{2}}(w_aw_a'+h_ah_a')^2-A^{-\frac{3}{2}}(1+w_aw_a''+h_ah_a'')\right|\\&\leq\frac{3}{2}A^{-\frac{5}{2}}(w_a^2+h_a^2)+A^{-\frac{3}{2}}(1+(w_a^2+h_a^2)^{\frac{1}{2}}(w_a''^2+h_a''^2)^{\frac{1}{2}}).\end{aligned}$$

The two principal curvatures of S_a with induced Euclidean metric are $-\dfrac{h_a'}{aw_a}$ and $a^{-1}(w_a''^2+h_a''^2)^{\frac{1}{2}}$ ([10] p.162, (10)), hence by Remark 1.9, $|(A^{-\frac{1}{2}})''|=O(1)$. $\square$

From now on, we will drop the subscript a and denote w_a by w, h_a by h, ϕ_a by ϕ and l_a by l. We also denote ds^2 to be the metric on S_a induced from g.

Lemma 1.15. *The Gaussian curvature K of (S_a, ds^2) is positive for sufficiently large a. In particular, there exists a unique isometric embedding of (S_a, ds^2) into $(\mathbb{R}^3, \delta)$ for sufficiently large a.*

Proof. Let $d\overline{s}^2$ and ds^2 be the metrics on S_a induced by δ and g respectively.

$$\begin{cases} d\overline{s}^2 & = a^2(d\varphi^2 + w^2 d\theta^2) = \overline{E}d\varphi^2 + \overline{G}d\theta^2, \\ ds^2 & = \phi^4 d\overline{s}^2 = Ed\varphi^2 + Gd\theta^2, \end{cases}$$

implies

$$\begin{cases} E = \overline{E} + O(a), E_\varphi = \overline{E}_\varphi + O(a), E_{\varphi\varphi} = \overline{E}_{\varphi\varphi} + O(a) \\ E_\theta = \overline{E}_\theta + O(a), E_{\theta\theta} = \overline{E}_{\theta\theta} + O(a). \end{cases}$$

Similar result holds for G. By the formula $K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_\theta}{\sqrt{EG}}\right)_\theta + \left(\frac{G_\phi}{\sqrt{EG}}\right)_\phi\right)$ and the corresponding formula for $\overline{K}$, one can get $K = \overline{K} + O\left(a^{-3}\right)$. Hence the lemma holds. $\square$

Now let us compute the mean curvature of a revolution surface in $(\mathbb{R}^3, \delta)$.

Lemma 1.16. *For a smooth revolution surface S in $(\mathbb{R}^3, \delta)$ parametrized by*

$$(au(\varphi)\cos\theta, au(\varphi)\sin\theta, av(\varphi)), \quad 0 < \varphi < l, 0 < \theta < 2\pi,$$

its mean curvature $\overline{H}$ with respect to δ is

$$\overline{H} = \frac{u''}{aTv'} - \frac{T'u'}{aT^2v'} - \frac{v'}{aTu} \quad \text{where } T = \sqrt{u'^2 + v'^2}.$$

Proof. The mean curvature $\overline{H}$ of S with respect to δ is computed to be

$$\overline{H} = \frac{v'u'' - u'v''}{aT^3} - \frac{v'}{aTu}.$$

Differentiating T^2 gives $u'u'' + v'v'' = TT'$. This implies

$$v'u'' - u'v'' = v'u'' + \frac{u'^2u'' - u'TT'}{v'} = \frac{(u'^2 + v'^2)u'' - u'TT'}{v'} = \frac{T^2u'' - TT'u'}{v'}.$$

So we have $\overline{H} = \frac{u''}{aTv'} - \frac{T'u'}{aT^2v'} - \frac{v'}{aTu}$. $\square$

Lemma 1.17. *The mean curvature H of S_a with respect to g is*

$$H = \frac{w''}{a\phi^2 h'} - \frac{h'}{a\phi^2 w} + 4\phi^{-3} n(\phi) \tag{1.20}$$

where n is the outward unit normal vector of S_a with respect to δ.

Proof. By Lemma 1.16, the mean curvature of S_a with respect to δ is $\overline{H} = \frac{w''}{ah'} - \frac{h'}{aw}$. The mean curvature H of S_a with respect to g is ([31], p. 72) $H = \phi^{-2}\left(\overline{H} + 4\phi^{-1} n(\phi)\right)$. The result follows. □

Lemma 1.18. *For sufficiently large a, there is an isometric embedding of (S_a, ds^2) into $(\mathbb{R}^3, \delta)$ which is given by*

$$(x^1, x^2, x^3) = (au(\varphi)\cos\theta, au(\varphi)\sin\theta, av(\varphi)), \quad \varphi \in [0, l], \theta \in [0, 2\pi] \tag{1.21}$$

where

$$\begin{cases} u = \phi^2 w, \quad v' = \phi^2 h' \left(1 - \frac{2\phi' w w'}{h'^2} + O\left(a^{-2}\right)\right), \\ u'^2 + v'^2 = \phi^4. \end{cases}$$

Proof. The existence has already been proved in Lemma 1.15.

In (φ, θ) coordinates, the metric on S_a induced by g can be written as:

$$ds^2 = a^2\phi^4 d\varphi^2 + a^2\phi^4 w^2 d\theta^2. \tag{1.22}$$

We can regard (S_a, ds^2) as $\mathbb{S}^2$, the sphere with the metric ds^2. Now we want to find two functions u, v such that the surface written as the form (1.21) is an embedded surface S_a^e of S_a into $(\mathbb{R}^3, \delta)$. First of all, the induced metric by the Euclidean metric on the surface which is of the form (1.21) can be written as:

$$ds_e^2 = a^2\left(u'^2 + v'^2\right) d\varphi^2 + a^2 u^2 d\theta^2.$$

Comparing this with (1.22), one can choose

$$u = \phi^2 w. \tag{1.23}$$

Consider

$$\begin{aligned}
\phi^4 - u'^2 = \phi^2(\phi^2 - (2\phi' w + \phi w')^2) &= \phi^2(\phi^2(w'^2 + h'^2) - (2\phi' w + \phi w')^2) \\
&= \phi^2(\phi^2 h'^2 - 4\phi\phi' w w' - 4\phi'^2 w^2) \\
&= \phi^4 h'^2 \left(1 - \frac{4\phi' w w'}{\phi h'^2} - \frac{4\phi'^2 w^2}{\phi^2 h'^2}\right).
\end{aligned} \tag{1.24}$$

By Lemma 1.13 and Lemma 1.14, the functions $\frac{\phi' w w'}{\phi h'^2}, \frac{\phi'^2 w^2}{\phi^2 h'^2}$ can be extended continuously on $[0, l]$ with $\frac{\phi' w w'}{\phi h'^2} = O(a^{-1}), \frac{\phi'^2 w^2}{\phi^2 h'^2} = O(a^{-2})$. So $1 - \frac{4\phi' w w'}{\phi h'^2} - \frac{4\phi'^2 w^2}{\phi^2 h'^2} > 0$ for sufficiently large a. For these a, we can take

$$v' = \phi^2 h' \left(1 - \frac{4\phi' w w'}{\phi h'^2} - \frac{4\phi'^2 w^2}{\phi^2 h'^2}\right)^{\frac{1}{2}}$$

so that $u'^2 + v'^2 = \phi^4$. Note that by (1.18), v' is an odd function for $\varphi \in [-l, l]$. By choosing an initial value, one can get an even function v. By the above argument, one has

$$v' = \phi^2 h' \left(1 - \frac{2\phi' w w'}{h'^2} + O\left(a^{-2}\right)\right).$$

From (1.23) and (1.24), near $\varphi = 0$, $u,\ v$ can be extended naturally to $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Since u is an odd function in φ , v is an even function in φ, and $u'^2 + v'^2 = T^2 > 0$, the generating curve in $\{x^2 = 0\}$ is symmetric with respect to x^3-axis, and is smooth at $\varphi = 0$. Similarly, it is also smooth at $\varphi = l$. Hence the revolution surface determined by the choice of $u,\ v$ as above, can be extended smoothly to a closed revolution surface, which is an embedded surface of S_a. This completes the proof of the lemma. □

1.2.3 Proof of Proposition 1.8

Now we are ready to prove Proposition 1.8.

Proof of Proposition 1.8. Let u, v be defined as in Lemma 1.18. Recall that

$$\begin{cases} u = \phi^2 w, v' = \phi^2 h' \left(1 - \frac{2\phi' w w'}{h'^2} + O\left(a^{-2}\right)\right), \\ u'^2 + v'^2 = \phi^4 = T^2 \quad \text{where } T = \phi^2. \end{cases} \tag{1.25}$$

By Lemma 1.14, we have

$$\begin{cases} T' = 2\phi' + O\left(a^{-2}\right), u' = \phi^2 w' + O\left(a^{-1}\right), \\ u'' = \phi^2 w'' + 4\phi' w' + 2\phi'' w + O\left(a^{-2}\right). \end{cases} \tag{1.26}$$

By Lemma 1.16 and Lemma 1.18,

$$H_0 = \frac{u''}{aTv'} - \frac{T'u'}{aT^2v'} - \frac{v'}{aTu}. \tag{1.27}$$

Combining with Lemma 1.17,

$$H_0 - H = \left(\frac{u''}{aTv'} - \frac{w''}{a\phi^2 h'}\right) - \frac{T'u'}{aT^2v'} - \left(\frac{v'}{aTu} - \frac{h'}{a\phi^2 w}\right) - 4\phi^{-3}n(\phi). \tag{1.28}$$

Using (1.25) and (1.26),

$$\begin{aligned} \frac{u''}{aTv'} - \frac{w''}{a\phi^2 h'} &= \frac{w''}{a\phi^2 h'} + \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} + \frac{2\phi' w w' w''}{ah'^3} - \frac{w''}{a\phi^2 h'} + O\left(a^{-3}\right) \\ &= \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} + \frac{2\phi' w w' w''}{ah'^3} + O\left(a^{-3}\right). \end{aligned} \tag{1.29}$$

By (1.25) and (1.26),

$$-\frac{T'u'}{aT^2v'} = -\frac{2\phi' w'}{ah'} + O\left(a^{-3}\right). \tag{1.30}$$

By (1.25),

$$-\frac{v'}{aTu} + \frac{h'}{a\phi^2 w} = -\frac{h'}{a\phi^2 w} + \frac{2\phi' w'}{ah'} + \frac{h'}{a\phi^2 w} + O\left(a^{-3}\right) = \frac{2\phi' w'}{ah'} + O\left(a^{-3}\right). \tag{1.31}$$

Summing (1.29), (1.30) and (1.31) and comparing with (1.28), we have

$$H_0 - H = \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} + \frac{2\phi' w w' w''}{ah'^3} - 4\phi^{-3}n(\phi) + O\left(a^{-3}\right).$$

As $w'w'' = -h'h''$ by (1.17), so

$$H_0 - H = \frac{4\phi' w'}{ah'} + \frac{2\phi'' w}{ah'} - \frac{2\phi' w h''}{ah'^2} - 4\phi^{-3}n(\phi) + O\left(a^{-3}\right).$$

Denote the Euclidean area element of S_a by $d\sigma_0$, the area element of (S_a, ds^2) by $d\sigma$. Note that $H_0 - H = O\left(a^{-2}\right)$, $d\sigma - d\sigma_0 = O\left(a^{-1}\right) d\sigma_0$ and $\displaystyle\int_{S_a} d\sigma_0 = O\left(a^2\right)$. To prove the result, it suffices to show

$$\lim_{a\to\infty} \frac{1}{8\pi} \int_{S_a} \left(H_0 - H\right) d\sigma_0 = m.$$

The Euclidean area element is computed to be $d\sigma_0 = a^2 w d\varphi d\theta$. By (1.19) and Lemma 1.13,

$$\begin{aligned}\int_{S_a}\left(\frac{4\phi' w'}{ah'}+\frac{2\phi'' w}{ah'}-\frac{2\phi' w h''}{ah'^2}\right)d\sigma_0 =&2\pi a\int_0^l\left(\frac{4\phi' w w'}{h'}+\frac{2\phi'' w^2}{h'}-\frac{2\phi' w^2 h''}{h'^2}\right)d\varphi\\ =&2\pi a\int_0^l\frac{d}{d\varphi}\left(\frac{2\phi' w^2}{h'}\right)d\varphi\\ =&0.\end{aligned}$$

Since the norm of the Euclidean gradient of ϕ has $|\nabla_0\phi| = O(r^{-2})$, one has $n(\phi) = O(a^{-2})$. So

$$\begin{aligned}\frac{1}{8\pi}\int_{S_a}(H_0 - H)\, d\sigma_0 &= -\frac{1}{2\pi}\int_{S_a}\phi^{-3} n(\phi) d\sigma_0 + O\left(a^{-1}\right)\\ &= -\frac{1}{2\pi}\int_{S_a} n(\phi) d\sigma_0 + O\left(a^{-1}\right).\end{aligned}$$

By the result of [3] (Proposition 4.1), the definition of the ADM mass of N can be taken as

$$\lim_{a\to\infty}\frac{1}{16\pi}\int_{S_a}\sum_{i,j}(g_{ij,i} - g_{ii,j})n^j d\sigma_0 = m. \tag{1.32}$$

where n is the unit outward normal of S_a with respect to δ. By a direct computation,

$$\sum_{i,j}(g_{ij,i} - g_{ii,j})n^j = -8\phi^3\sum_j n^j\frac{\partial\phi}{\partial x^j} = -8n(\phi) + O\left(a^{-3}\right). \tag{1.33}$$

Combining (1.32) and (1.33), we have

$$m = -\lim_{a\to\infty}\frac{1}{2\pi}\int_{S_a} n(\phi) d\sigma_0.$$

Therefore

$$\lim_{a\to\infty}\frac{1}{8\pi}\int_{S_a}(H_0 - H) d\sigma = \lim_{a\to\infty}\frac{1}{8\pi}\int_{S_a}(H_0 - H) d\sigma_0 = m.$$

We are done. □

Chapter 2

Quasi-local mass in AH manifolds

It is known that in an asymptotically flat manifold, the Brown-York quasi-local mass of the coordinate spheres will converge to the ADM mass of the manifold [12, 36, 11]. In this chapter, we will show an analogous result for asymptotically hyperbolic (AH) manifolds.

2.1 Asymptotically hyperbolic (AH) manifolds

First we give the meanings of mass of an AH manifold and quasi-local mass. In this chapter, all manifolds are assumed to be connected and orientable.

We will follow X. D. Wang [40] to define asymptotically hyperbolic manifolds as follows:

Definition 2.1. *A complete noncompact Riemannian manifold (M^n, g) is said to be asymptotically hyperbolic (AH) if M is the interior of a compact manifold $\overline{M}$ with boundary $\partial\overline{M}$ such that:*

(i) there is a smooth function r on $\overline{M}$ with $r > 0$ on M and $r = 0$ on $\partial\overline{M}$ such that $\overline{g} = r^2 g$ extends as a smooth Riemannian metric on $\overline{M}$;

(ii) $|dr|_{\overline{g}} = 1$ on $\partial\overline{M}$;

(iii) $\partial\overline{M}$ *is the standard unit sphere* $\mathbb{S}^{n-1}$*;*

(iv) on a collar neighborhood of $\partial\overline{M}$*,*

$$g = \sinh^{-2}(r)(dr^2 + g_r),$$

with g_r *being an* r*-dependent family of metrics on* $\mathbb{S}^{n-1}$ *satisfying*

$$g_r = g_0 + \frac{r^n}{n}h + e,$$

where g_0 *is the standard metric,* h *is a smooth symmetric 2-tensor on* $\mathbb{S}^{n-1}$ *and* e *is of order* $O(r^{n+1})$*, and the asymptotic expansion can be differentiated twice.*

Note that the definition is not as general as that in [9], see also [42]. In [40], the following positive mass theorem was proved by Wang (see also [1, 9, 42])

Theorem 2.2. *[40, Theorem 2.5] If* (M^n, g) *is spin, asymptotically hyperbolic and the scalar curvature* $R \geq -n(n-1)$*, then*

$$\int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h)d\mu_{g_0} \geq \left| \int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h)xd\mu_{g_0} \right|.$$

Moreover equality holds if and only if (M, g) *is isometric to the hyperbolic space* $\mathbb{H}^n$*.*

We only consider the case that $n = 3$, the theorem implies that if M is not isometric to the hyperbolic space, then the vector

$$\Upsilon = \left(\int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h)d\mu_{g_0}, \int_{\mathbb{S}^{n-1}} \mathrm{tr}_{g_0}(h)xd\mu_{g_0} \right)$$

is a future directed timelike vector in $\mathbb{R}^{3,1}$, the Minkowski space. We may consider Υ as the mass integral for the AH manifold.

2.2 Quasi-local mass integral of AH manifolds

We introduce the following quasi-local mass integral for a compact manifold with boundary, similar to the Brown-York mass. Let (Ω, g) be a three dimensional compact manifold with smooth boundary Σ. Assume Σ is homeomorphic to the standard sphere $\mathbb{S}^2$ such that the mean curvature of Σ is positive and the Gaussian curvature of Σ is larger than -1. Then Σ can be isometrically embedded into the hyperbolic space $\mathbb{H}^3$ by a result of Pogorelov [28] and the embedding is unique up to an isometry of $\mathbb{H}^3$. Consider $\mathbb{H}^3$ as the hyperboloid in $\mathbb{R}^{3,1}$

$$\mathbb{H}^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^{3,1} : (x^0)^2 - \sum_{i=1}^{3}(x^i)^2 = 1, x^0 > 0\}.$$

Then the quasi-local mass integral of Ω is defined as:

$$\int_\Sigma (H_0 - H)X$$

where H_0 is the mean curvature of Σ in $\mathbb{H}^3$ and X is the position vector in $\mathbb{R}^{3,1}$.

The motivation of this definition is as follows. In [38], M. T. Wang and Yau proved that if the scalar curvature of Ω satisfies $R \geq -6$, then there is a future time like vector W such that

$$\int_\Sigma (H_0 - H)W$$

is a future directed non-spacelike vector. W is obtained by solving a backward parabolic equation with a prescribed data at infinity and is not very explicit. Later in [35], Shi and Tam proved that if $B_o(R_1)$ and $B_o(R_2)$ are two geodesic balls in $\mathbb{H}^3$ such that $B_o(R_1)$ is contained in the interior of Σ in $\mathbb{H}^3$ and Σ is contained in $B_o(R_2)$, where $o = (1, 0, 0, 0) \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$, then the result of Wang-Yau is true for $W(x^0, x^1, x^2, x^3) = (\alpha x^0, x^1, x^2, x^3)$ with

$$\alpha = \coth R_1 + \frac{1}{\sinh R_1}\left(\frac{\sinh^2 R_2}{\sinh^2 R_1} - 1\right)^{\frac{1}{2}}.$$

Hence W is close to the position vector. It is an open question whether W can be chosen to be the position vector.

In this chapter, we consider AH manifolds with the following condition (with the notations as in Definition 2.1):

Assumption A: $\nabla_{\mathbb{S}^{n-1}}e, \nabla^2_{\mathbb{S}^{n-1}}e, \nabla^3_{\mathbb{S}^{n-1}}e, \nabla^4_{\mathbb{S}^{n-1}}e$ with respect to g_0 and $\frac{\partial e}{\partial r}$ are of order $O(r^n)$.

Let $S_a = \{r = a\} \subset (M, g)$ and let H to be its mean curvature. We identify S_r as the standard sphere $\mathbb{S}^2$ with metric γ_r induced from g. Then for r small, the Gaussian curvature of (S_r, γ_r) is positive where γ_r is the induced metric of g.

Our main result is the following:

Theorem 2.3. *[20] Let (M, g) be a three-dimensional asymptotically hyperbolic manifold satisfying Assumption A. For all r sufficiently small, there exists an isometric embedding $X^{(r)} : S_r \to \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ such that*

$$\lim_{r\to 0}\int_{S_r}(H_0 - H)X^{(r)}d\mu_{\gamma_r} = \frac{1}{2}\left(\int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)d\mu_{g_0}, \int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)x d\mu_{g_0}\right)$$

where H_0 is the mean curvature of $X^{(r)}(S_r)$ in $\mathbb{H}^3$.

Remark 2.4. *From the proof of Theorem 2.3, $X^{(r)}$ in the theorem can be chosen by applying an isometry of $\mathbb{H}^3$ fixing o (i.e. $O(3)$) on $\widetilde{X}^{(r)}$, where $\widetilde{X}^{(r)}$ is an embedding of S_r (for small r) such that o is the center of a largest geodesic sphere contained in the interior of $\widetilde{X}^{(r)}(S_r)$ (or a smallest geodesic sphere containing $\widetilde{X}^{(r)}(S_r)$ in its interior).*

By applying Theorem 2.2 to our result, we have

Corollary 2.5. *Let (M, g) be a three-dimensional asymptotically hyperbolic manifold satisfying Assumption A with the scalar curvature $R \geq -6$, if $Y^{(r)} : S_r \to \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ is an isometric embedding such that o is the center of a largest geodesic sphere contained in the interior of $Y^{(r)}(S_r)$ (or a smallest geodesic sphere containing $Y^{(r)}(S_r)$ in its interior), then for sufficiently small r, the vector*

$$\int_{S_r}(H_0 - H)Y^{(r)}d\mu_{\gamma_r}$$

is either zero or is future-directed timelike. If (M, g) is not isometric to $\mathbb{H}^3$, then this vector is always non-zero for sufficiently small r.

Let us give the outline of the coming sections. In Section 2.2.1, we will establish some estimates for the various curvatures of S_r and its embedding in the hyperbolic space. In Section 2.2.2, we will describe some basic results in hyperbolic geometry concerning the radii of the smallest geodesic sphere enclosing a given convex surface and of the largest geodesic sphere enclosed by it. In Section 2.2.3, we will normalize the isometric embedding of S_r into the hyperbolic space so that the image of the isometric embedding of S_r is close to a geodesic sphere in the hyperbolic space. We then prove the main results in Section 2.2.4.

2.2.1 Curvature estimates

In this section, we always assume (M^3, g) is a three dimensional AH manifold as in Definition 2.1 such that Assumption A is satisfied. Using the notations in Definition 2.1, let $S_a = \{r = a\} \subset M$. We want to obtain some curvature estimates for S_r which will be used in the proof of the main result. First we will estimate the intrinsic scalar curvature R which is twice the Gaussian curvature of S_r with the metric γ_r induced by g.

Lemma 2.6. *The scalar curvature R of S_r with respect to the induced metric from g is given by*

$$R = 2\sinh^2 r + O(r^5).$$

Proof. Recall that $g_r = g_0 + \frac{r^3}{3}h + e$. Then $\gamma_r = \sinh^{-2}(r)g_r$ is the induced metric on S_r from g. Let R and $\widetilde{R}$ be the scalar curvature of S_r with respect to the metric γ_r and g_r respectively. It is easy to see that $R = \sinh^2(r)\widetilde{R}$. We claim that

$$\widetilde{R} = 2 + O(r^3). \tag{2.1}$$

The result immediately follows from this claim.

To prove the claim, let $\{y^i\}_{i=1}^2$ be the local coordinates on the lower hemisphere (say) of $\mathbb{S}^2$ induced by the stereographic projection from the north pole to the plane. Let $\widetilde{g}_{ij} = g_r(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}), g_{ij} = g_0(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})$ and $\widetilde{\Gamma}^k_{ij}, \Gamma^k_{ij}$ be the Christoffel symbols with respect to $\widetilde{g}_{ij}$ and g_{ij} respectively. Let $\widetilde{g}^{ij}$ and g^{ij} be the inverse of $\widetilde{g}_{ij}$ and g_{ij} respectively. Then

$$\widetilde{R} = \sum_{j,k,l} \widetilde{g}^{jk} \widetilde{R}^l_{ljk} \text{ where } \widetilde{R}^l_{ijk} = \partial_i \widetilde{\Gamma}^l_{jk} - \partial_j \widetilde{\Gamma}^l_{ki} + \sum_p \widetilde{\Gamma}^p_{jk} \widetilde{\Gamma}^l_{ip} - \sum_p \widetilde{\Gamma}^p_{ik} \widetilde{\Gamma}^l_{jp}, \tag{2.2}$$

and

$$2 = \sum_{j,k,l} g^{jk} R^l_{ljk} \text{ where } R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ki} + \sum_p \Gamma^p_{jk} \Gamma^l_{ip} - \sum_p \Gamma^p_{ik} \Gamma^l_{jp}. \tag{2.3}$$

Assumption A implies that

$$|\widetilde{g}_{ij} - g_{ij}| = O(r^3), |\widetilde{g}_{ij,k} - g_{ij,k}| = O(r^3) \text{ and } |\widetilde{g}_{ij,kl} - g_{ij,kl}| = O(r^3),$$

where $g_{ij,k} = \frac{\partial g_{ij}}{\partial y^k}$ etc. Hence

$$\widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij} = O(r^3) \text{ and } \partial_i \widetilde{\Gamma}^l_{jk} - \partial_i \Gamma^l_{jk} = O(r^3). \tag{2.4}$$

In view of (2.2) and (2.3), these imply that $\widetilde{R}^l_{ijk} - R^l_{ijk} = O(r^3)$ and hence $\widetilde{R} - 2 = O(r^3)$. We conclude that (2.1) is true. This completes the proof of the lemma. □

Next, we want to estimate the mean curvature H of S_r with respect to g.

Lemma 2.7. *If (M, g) is asymptotically hyperbolic satisfying Assumption A, then the mean curvature of S_r is*

$$H = 2\cosh r - \frac{1}{2} r^3 \mathrm{tr}_{g_0} h + O(r^4).$$

Proof. Let $\{e_j\}_{j=1}^2$ be a local orthonormal frame on $(\mathbb{S}^2, g_0)$. The outer unit normal of S_r is $\nu = -\sinh r \frac{\partial}{\partial r}$. Denote $g(e_i, e_j)$ by g_{ij} and $g_r(e_i, e_j)$ by σ_{ij}, then

$$\begin{aligned} H &= \nu\left(\log \sqrt{\det(g_{ij})}\right) \\ &= -\sinh r \frac{\partial}{\partial r}\left(\log\left(\sinh^{-2} r \sqrt{\det(\sigma_{ij})}\right)\right) \\ &= 2\cosh r - \frac{1}{2} \frac{\sinh r}{\sqrt{\det(\sigma_{ij})}} \frac{\partial}{\partial r} \det(\sigma_{ij}). \end{aligned}$$

It is easy to see that $\det(\sigma_{ij}) = 1 + \frac{r^3}{3}\mathrm{tr}_{g_0}h + O(r^4)$ and by the condition $\frac{\partial e}{\partial r} = O(r^3)$, that $\frac{\partial}{\partial r}\det(\sigma_{ij}) = r^2\mathrm{tr}_{g_0}h + O(r^3)$. Combining these with the above calculation, we can get the result. □

By Lemma 2.6, for sufficiently small r, the Gaussian curvature K of (S_r, γ_r) is positive. Hence (S_r, γ_r) can be isometrically embedded into $\mathbb{H}^3$ which is unique up to an isometry in $\mathbb{H}^3$ by the results of Pogorelov [28]. Moreover, by the Gauss equation, for an orthonormal frame in S_r,

$$-1 + \chi_{11}\chi_{22} - \chi_{12}^2 = K > 0.$$

Hence the embedded surface which will be denoted by Σ_r is strictly convex. Let H_0 be the mean curvature of Σ_r, we want to estimate H_0 and compare it with H.

To estimate H_0, we will generalize a result on convex compact hypersurfaces in $\mathbb{R}^n$ of Li-Weinstein [22, Theorem 2] to compact hypersurfaces in $\mathbb{H}^n$.

Lemma 2.8. *Suppose Σ is a closed convex hypersurface in $\mathbb{H}^n$, $n \geq 3$. If the scalar curvature R of Σ satisfies $R + (n-2)(n-3) > 0$, then its mean curvature H_0 satisfies the inequality*

$$H_0^2 \leq \max_{\Sigma}\left(\frac{2\widehat{R}^2 - 2(n-1)\widehat{R} - \Delta R}{R + (n-2)(n-3)}\right)$$

where $\widehat{R} = R + (n-1)(n-2)$ and Δ is the Laplacian on Σ.

Proof. We basically follow the ideas from [22]. Let χ be the second fundamental form of $\Sigma \subset \mathbb{H}^n$. Let $p \in \Sigma$ be such that $H_0(p) = \max_{\Sigma} H_0$. Let $\{x^j\}_{j=1}^{n-1}$ be a normal coordinates of Σ around p so that $\chi_{ij} = \lambda_i\delta_{ij}$ at p. Then at p, $H_{0;ij}$ is negative semi definite. Here we use $S_{;k}$ to denote the covariant derivative of S on Σ with respect to the induced metric. Since χ_{ij} is positive, at p we have,

$$H_0\Delta H_0 = \left(\sum_i \lambda_i\right)\left(\sum_i H_{0;ii}\right) \leq \sum_i \lambda_i H_{0;ii}. \tag{2.5}$$

All sums here will have indices from 1 to $n-1$. Since $\mathbb{H}^n$ has constant curvature, the Codazzi equation implies

$$\chi_{ij;k} - \chi_{ik;j} = 0. \tag{2.6}$$

By the Gauss equation, we have

$$R + (n-1)(n-2) = H_0^2 - |\chi|^2. \tag{2.7}$$

Let R_{ijkl} be the intrinsic curvature tensor of Σ. At p,

$$\begin{aligned}
\Delta R &= 2H_0 \Delta H_0 + 2|\nabla H_0|^2 - 2|\nabla \chi|^2 - 2\sum_{i,k} \lambda_i \chi_{ii;kk} \\
&\le 2\sum_{i,k} \lambda_i \left(\chi_{kk;ii} - \chi_{ii;kk}\right) \quad \text{(by (2.5) and } \nabla H_0 = 0) \\
&= 2\sum_{i,k} \lambda_i \left(\chi_{kk;ii} - \chi_{ki;ik}\right) \quad \text{(by (2.6))} \\
&= 2\sum_{i,k,m} \chi_{ij}(R_{kikm}\chi_{mi} + R_{kiim}\chi_{km}) \quad \text{(by Ricci identity and (2.6))} \\
&= 2\sum_{i,k} R_{kiik}(-\lambda_i^2 + \lambda_i\lambda_k) \\
&= 2\sum_{i,k} (-1 + \lambda_k\lambda_i)(-\lambda_i^2 + \lambda_i\lambda_k) \quad \text{(by the Gauss equation)} \\
&= 2\left((n-1)|\chi|^2 - H_0 \sum_i \lambda_i^3 - H_0^2 + |\chi|^4\right).
\end{aligned}$$

By [22, Lemma 2], since $\lambda_i > 0$,

$$-2\sum_i \lambda_i^3 \le \left(\sum_i \lambda_i\right)^3 - 3\left(\sum_i \lambda_i^2\right)\left(\sum_i \lambda_i\right) = H_0^3 - 3|\chi|^2 H_0.$$

Plugging this into the above and use (2.7), at p,

$$\begin{aligned}
\Delta R &\le 2(n-1)|\chi|^2 + 3\widehat{R} H_0^2 - 2H_0^4 + 2|\chi|^4 - 2H_0^2 \\
&= 2(n-1)(H_0^2 - \widehat{R}) + 3\widehat{R} H_0^2 - 2H_0^4 + 2(H_0^2 - \widehat{R})^2 - 2H_0^2 \\
&= -(\widehat{R} - 2(n-2))H_0^2 - 2(n-1)\widehat{R} + 2\widehat{R}^2.
\end{aligned}$$

From this it is easy to see that the lemma is true. $\square$

Applying the previous lemma to Σ_r which is the embedded image of (S_r, γ_r), we have:

Corollary 2.9. *With the same assumptions and notations as in Lemma 2.6, for sufficiently small r, the mean curvature H_0 of Σ_r in $\mathbb{H}^3$ satisfies*

$$H_0^2 \leq \max_{S_r}(2R + 4 - \frac{\Delta R}{R})$$

where Δ is the Laplacian on S_r under the induced metric, $R = 2K$ and K is the Gaussian curvature of S_r.

We now estimate H_0.

Lemma 2.10. *The mean curvature H_0 of Σ_r in $\mathbb{H}^3$ is given by*

$$H_0 = 2\cosh r + O(r^5).$$

Proof. By the Gauss equation, $2\widehat{R} \leq \widehat{R} + |\chi|^2 = H_0^2$ where $\widehat{R} = R + 2$ and χ is χ is the second fundamental form of the embedded S_r. So by combining Lemma 2.6 and Corollary 2.9, we have

$$4\cosh^2 r + O(r^5) \leq H_0^2 \leq 4\cosh^2 r + \max_{S_r}\left|\frac{\Delta R}{R}\right| + O(r^5).$$

The proof would be completed if we can show that $\frac{\Delta R}{R} = O(r^5)$. The proof is analogous to that of Lemma 2.6. Using the notations in the proof of Lemma 2.6, it is easy to see that

$$\frac{\Delta R}{R} = \frac{\sinh^4 r}{R}\Delta_{g_r}\widetilde{R} \tag{2.8}$$

where $\widetilde{R}$ is the scalar curvature with respect to g_r. Using Assumption A, we have

$$|\partial^{(k)}\widetilde{\Gamma}_{ij}^l - \partial^{(k)}\Gamma_{ij}^l| = O(r^3) \text{ for } k = 0, 1, 2, 3,$$

with respect to the coordinates $\{y^i\}_{i=1}^2$. Together with (2.2) and (2.3), we conclude that $\partial_i\widetilde{R} - \partial_i R = O(r^3)$ and $\partial_{ij}^2\widetilde{R} - \partial_{ij}^2 R = O(r^3)$. Hence

$$\Delta_{g_r}\widetilde{R} - \Delta_{g_0}R_0 = O(r^3).$$

As $R_0 = 2$ is a constant, by (2.8) and Lemma 2.6, the result follows. □

Combining Lemma 2.7 and Lemma 2.10, we have

Corollary 2.11. *On S_r,*

$$H_0 - H = \frac{1}{2} r^3 \mathrm{tr}_{g_0} h + O(r^4).$$

2.2.2 Inscribed and circumscribed geodesic spheres

It is well known that a compact convex hypersurface Σ in $\mathbb{R}^n$ can contain and be contained in spheres with radius depending only on the upper and lower bound of principal curvatures λ_i. In this section, we will describe the corresponding results in $\mathbb{H}^n$, which will be used later. We will sketch the proofs for the sake of completeness whenever we could not locate a reference. We only consider the case $n = 3$. The general case is similar. The following is a direct consequence of a result of Ralph Howard [17, Theorem 4.5].

Proposition 2.12. *Let Σ be a compact convex surface in $\mathbb{H}^3$ and $\coth b = \max_{x\in\Sigma} \lambda_i(x) \geq \min_{x\in\Sigma} \lambda_i(x) > 1$, then there is a geodesic sphere of radius b which is contained in the interior of Σ.*

Proof. By [17, Theorem 4.5], since $\lambda_i > 1$ on Σ, the largest radius (rolling radius) of geodesic balls which can roll inside Σ is equal to the focal distance of Σ.

We claim that the focal distance of Σ in $\mathbb{H}^3$ is equal to

$$\min_{x\in\Sigma} \{\rho : \coth \rho = \lambda_i(x), i = 1, 2\}.$$

The result then immediately follows.

To prove the claim, we use the following characterization of the focal distance in terms of Jacobi field ([17] p. 474). For $p \in \Sigma$, a Σ-adapted Jacobi field $V(s)$ along the inward-pointing arc-length parametrized geodesic $\gamma(s)$ starting from p is one which satisfies

$$V'' + R(V, \gamma')\gamma' = 0,\ V(0) \in T_p\Sigma,\ V'(0) = -A(V(0))$$

where $V^{(k)} = \nabla^{(k)}_{\gamma'} V$ and A is the shape operator. For such a $V \neq 0$, $\gamma(l)$ is a focal point of Σ along γ if $V(l) = 0$. If $\gamma(l)$ is the first focal point along γ, the focal distance at p is then l and the focal distance of Σ is the minimum of the focal distances among all $p \in \Sigma$.

Now, if e is an unit eigenvector of A_p with eigenvalue λ, we can then parallel translate e along $\gamma(s)$ to form $e(s)$. We define $V(s) = (\cosh s - \lambda \sinh s)e(s)$. Then

$$V'' = V \quad \text{and} \quad R(V, \gamma')\gamma' = -\langle \gamma', \gamma' \rangle V + \langle V, \gamma' \rangle \gamma' = -V.$$

Thus $V'' + R(V, \gamma')\gamma' = 0$. Also, $V'(0) = -\lambda e(0) = -A(e)$, $V(0) \in T_p\Sigma$. i.e. V is a Σ-Jacobi field. As $V(r) = 0$ where $\coth r = \lambda$, $\gamma(r)$ is a focal point.

Conversely, if $\gamma(l)$ is a focal point with the corresponding Σ-adapted Jacobi field $V(s)$. Then by the Jacobi field equation, $\langle V, \gamma' \rangle'' = 0$. The conditions of V implies $\langle V(0), \gamma'(0) \rangle = 0 = \langle V(l), \gamma'(l) \rangle$, so we have

$$\langle V, \gamma' \rangle \equiv 0.$$

Then

$$0 = V'' + R(V, \gamma')\gamma' = V'' - (\langle \gamma', \gamma' \rangle V - \langle V, \gamma' \rangle \gamma') = V'' - V.$$

Let $\{e_i\}_{i=1}^2$ be the unit eigenvectors of A_p with eigenvalues λ_i, we can then parallel translate e_i along $\gamma(s)$ to form $e_i(s)$. Let $V(s) = \sum_{i=1}^{2} f_i(s) e_i(s)$, then the above equation implies

$$f_i'' = f_i \quad \text{or,} \quad f_i = a_i \sinh s + b_i \cosh s.$$

But then $V'(0) = \sum_{i=1}^{2} a_i e_i = -A(V(0)) = -\sum_{i=1}^{2} b_i \lambda_i e_i$. So $a_i = -b_i \lambda_i$ for all i. Finally $V(l) = 0$ implies $b_i \cosh l = b_i \lambda_i \sinh l$. Therefore we have either $b_i = 0$ or $\coth l = \lambda_i$. We conclude that V is of the form $V(s) = (\cosh s - \lambda_i \sinh s)e(s)$ for some i and for some parallel $e(s)$. From this we can see that the claim is true. $\square$

For circumscribed geodesic spheres of Σ, we have the following:

Proposition 2.13. *Let Σ be a closed convex surface in $\mathbb{H}^3$ with $\lambda_i > \coth a > 1$ on Σ, then there is a geodesic sphere of radius a which contains Σ in its interior.*

Since we cannot find an explicit reference for this, we will give more details of the proof. We use the idea of Andrejs Treibergs [37] to give a proof. To show this, we need the following lemma about convex curves on $\mathbb{H}^2$ which is an extension of Schur's theorem for plane curves.

Lemma 2.14. *Let α and β be two curves in $\mathbb{H}^2$ with same length l parametrized by arc length. Suppose let γ be the geodesic from $\alpha(0)$, $\alpha(l)$ and σ be the geodesic from $\beta(0)$ to $\beta(l)$. Suppose α and γ bounds a geodesically convex region, and β, σ bounds a geodesically convex region. Suppose the geodesic curvature k of α is larger than the geodesic curvature $\widetilde{k}$ of β which are assumed to be positive. Then length of γ is less than the length of σ.*

Proof. Let us use the right half plane model for $\mathbb{H}^2$:

$$\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 |\ x > 0\}$$

with metric $ds^2 = \frac{dx^2+dy^2}{x^2}$. We may assume that γ is given by $\gamma(t) = (t, c)$, $a \le t \le b$ and c is a constant. We also assume that α is below γ. That is, if $\alpha(s) = (x(s), y(s))$, then $y(s) \le c$. We may assume that α touches the geodesic (t, c') for some c' at $\alpha(s_0)$ some $0 < s_0 < l$. Then α lies between the geodesics $y = c$ and $y = c'$. Move β such that $\beta(s_0) = \alpha(s_0)$, $\beta(s_0)$ touches $y = c'$ at $\beta(s_0)$ and β lies above $y = c'$; i.e., β is in the region $y \ge c'$.

Let $\alpha(s) = (x(s), y(s))$ and $\beta(s) = (\widetilde{x}(s), \widetilde{y}(s))$. Let $\theta(s)$ be the oriented angle from the tangent of the geodesic $(t, y(s))$ to $\alpha'(s)$. Define $\widetilde{\theta}(s)$ for β similarly so that $\theta(s_0) = \widetilde{\theta}(s_0) = 0$.

Note that for any $l > s > s' > s_0$, $y(s) \ne y(s')$, otherwise the curve $(t, y(s))$ is part of α which is a geodesic. This is impossible, because $k > 0$. Hence y is increasing in (s_0, l). So

$$x' = x\cos\theta, y' = x\sin\theta. \tag{2.9}$$

Hence $\sin\theta \geq 0$. But for $s_0 < s < l$, if $\sin\theta(s) = 0$, then the geodesic $(t, y(s))$ is tangent to α, which is impossible because of convexity of the region bounded by α and γ. So $\sin\theta > 0$, there.

On the other hand, we have [10, p. 253]:

$$k = -\sin\theta + \theta'.$$

Hence $0 < \theta \leq \pi$ on (s_0, l). Similarly, we have

$$\widetilde{k} = -\sin\widetilde{\theta} + \widetilde{\theta}'.$$

Since $k > \widetilde{k}$ and $\theta(s_0) = \widetilde{\theta}(s_0) = 0$, so for $s > s_0$ near s_0, $\theta(s) > \widetilde{\theta}(s)$. Suppose there is a first $l > s_1 > s_0$ such that $\theta(s_1) = \widetilde{\theta}(s_1)$. Then at s_1,

$$k - \widetilde{k} = \theta'(s_1) - \widetilde{\theta}'(s_1) \leq 0.$$

This is impossible. Hence $0 \leq \widetilde{\theta}(s) \leq \theta(s) \leq \pi$ in (s_0, l).

Now

$$\log x(l) - \log x(s_0) = \int_{s_0}^{l} \frac{x'}{x} ds = \int_{s_0}^{l} \cos\theta(s) ds$$

and

$$\log \widetilde{x}(l) - \log \widetilde{x}(s_0) = \int_{s_0}^{l} \frac{\widetilde{x}'}{\widetilde{x}} ds = \int_{s_0}^{l} \cos\widetilde{\theta}(s) ds$$

Hence $\log \widetilde{x}(l) \geq \log x(l) = \log b$. Similarly, one can prove that $\log \widetilde{x}(0) \leq \log x(0) = \log c$. In particular, $\widetilde{x}(0) < \widetilde{x}(l)$. Now the length $L(\gamma)$ of γ is $\log b - \log c$. Hence $L(\gamma) \leq \log \widetilde{x}(l) - \log \widetilde{x}(0)$.

We claim that $\log \widetilde{x}(l) - \log \widetilde{x}(0) \leq L(\sigma)$. We may assume $\widetilde{y}(0) < \widetilde{y}(l)$. Then $\log \widetilde{x}(l) - \log \widetilde{x}(0)$ is the length of the geodesic $(t, \widetilde{y}(l))$, $\widetilde{x}(0) < t < \widetilde{x}(l)$. Then by the sine law in $\mathbb{H}^2$, we conclude that the claim is true. This completes the proof of the lemma. □

Lemma 2.15. *Let α be a closed geodesically convex curve in $\mathbb{H}^2$ with geodesic curvature $k_\alpha > r > 0$. Let β be a geodesic circle with geodesic curvature r. Suppose α and β are tangent at p such that α and β lie on the same side of the geodesic through p and tangent to α and β. Then α will lie inside β.*

Proof. We use the disk model for $\mathbb{H}^2$. We may assume that β is a Euclidean circle with center at the origin and with radius $a > 0$, say. We may also assume that $p = (0, -a)$ and β is parametrized by $(a\cos\theta, a\sin\theta)$, $-\pi \le \theta \le \pi$. It is easy to see that $\beta(\theta)$ is outside α near p, for $\theta \in (-\frac{\pi}{2} - \theta_0, -\frac{\pi}{2} + \theta_0) = I$ for some $\theta_0 > 0$. Suppose the lemma is not true. Then β will intersect α at some $\theta_1 \notin I$. Without loss of generality, we may assume that there is $\frac{\pi}{2} \ge \theta_1 \ge -\frac{\pi}{2} + \theta_0$, such that α and β intersects at $q = \beta(\theta_1)$ and $\beta(\theta)$ lies strictly outside α in $(-\frac{\pi}{2} + \theta_0, \theta_1)$. Then the length of β from p to q is strictly larger than the length of α from p to q by the Gauss-Bonnet theorem and the fact that $k_\alpha > r$. Then there is $\theta_1 > \theta_2 > -\frac{\pi}{2} + \theta_0$ such that the length of β from p to $u = \beta(\theta_2)$ is the same as the length of α from p to q. By Lemma 2.14, we conclude that $d(p, q) \le d(p, u)$. Since p, q, u are on the geodesic circle β, this is impossible by the cosine law in $\mathbb{H}^2$.

□

Proof of Proposition 2.13. Let $p \in \Sigma$. Let S be the geodesic sphere with radius a which is tangent to Σ at p with the same unit outward normal at p. Let P be any normal section. That is, P is the totally geodesic $\mathbb{H}^2$ which passes through p and contains the geodesic normal to Σ (and S) at p. Let $\gamma = P \cap \Sigma$ and $\beta = P \cap S$.

Since the principal curvature of Σ is larger than $\coth a$, γ is a closed convex curve in P with geodesic curvature larger than $\coth a$. β is a geodesic circle of radius a in P. By Lemma 2.15, γ lies inside β and hence is inside S. Since P is an arbitrary normal section, the result follows. □

2.2.3 Normalized embedding of (S_r, γ_r)

Let (M^3, g) be an AH manifold satisfying Assumption A. Let (S_r, γ_r) be as in Lemma 2.6. The isometric embedding of (S_r, γ_r) is unique up to an isometry of $\mathbb{H}^3$. In order to prove the main results, we have to normalize the embedding. As a first step, using Lemmas 2.6 and 2.9, we can apply Propositions 2.12 and 2.13 to obtain the following:

Lemma 2.16. *With the above assumptions and notations, we can find a positive constant C such that for each small r, if Σ_r is the isometric embedding of (S_r, γ_r) in $\mathbb{H}^3$, then there exist geodesic balls B_{in} and B_{out} with the same center and radii ρ_{in} and ρ_{out} respectively, such that B_{in} is in the interior of Σ_r, B_{out} contains Σ_r and ρ_{in}, ρ_{out} satisfy:*

$$\rho_{in} \geq \sigma - Cr^3, \rho_{out} \leq \sigma + Cr^3, \tag{2.10}$$

where $\sigma = \sigma(r) > 0$ is given by $\sinh \sigma = \frac{1}{\sinh r}$.

Proof. Let r be a fixed small number. Let $\lambda_j(x)$ be the principal curvatures of $x \in \Sigma_r$. By Lemmas 2.6 and 2.10 and the Gauss equation, it is easy to see that

$$\lambda_j = \cosh r + O(r^5). \tag{2.11}$$

Let $\coth \rho = \lambda_j$, then

$$\begin{aligned}\rho = \frac{1}{2}\log(\frac{\lambda_j + 1}{\lambda_j - 1}) = \frac{1}{2}\log(\frac{\cosh r + 1 + O(r^5)}{\cosh r - 1 + O(r^5)}) =& \frac{1}{2}\log(\frac{\cosh r + 1}{\cosh r - 1}) + O(r^3) \\ =& \sigma + O(r^3).\end{aligned}$$

From this and Propositions 2.12 and 2.13, it is easy to see the corollary is true. □

By Lemma 2.16, the first normalization of the embedding is to normalize such that the center of the geodesic balls in Lemma 2.16 is at a fixed point $o \in \mathbb{H}^3$. We will use geodesic polar coordinates (σ, y) with center at o, where σ is the geodesic distance from o and $y \in \mathbb{S}^2$ so that a point in $\mathbb{H}^2$ is of the form $\exp_o(\sigma y)$. The metric $g_{\mathbb{H}^2}$ is given by $d\sigma^2 + \sinh^2 \sigma \; g_0$ where g_0 is the standard metric on $\mathbb{S}^2$.

The isometric embedding $X^{(r)}$ is given by $X^{(r)}(x) = \exp_o(\sigma^{(r)}(x) y^{(r)}(x))$.

Lemma 2.17. *With the above notations, there exists a constant $C > 0$ such that for all r small enough,*

$$\left| d_{\mathbb{S}^2}(x_1, x_2) - d_{\mathbb{S}^2}\left(y^{(r)}(x_1), y^{(r)}(x_2)\right) \right| \leq Cr^3$$

for $x_1, x_2 \in \mathbb{S}^2$, where $d_{\mathbb{S}^2}$ is the distance on $\mathbb{S}^2$ with respect to the standard metric.

Proof. Let $x_1, x_2 \in \mathbb{S}^2$ and let $X^{(r)}$ as above so that the embedded image Σ_r lies between two concentric geodesic spheres $\partial B_o(R_1)$ and $\partial B_o(R_2)$ with center at o and with radii $R_1 > R_2$ such that $R_i = \sigma + O(r^3)$, $i = 1, 2$, and σ is given by $\sinh\sigma = \frac{1}{\sinh r}$, by Lemma 2.16. Here and below $O(r^k)$ will denote a quantity with absolute value bounded by Cr^k for some positive constant C independent of r and $x_1, x_2 \in \mathbb{S}^2$.

Let $l(x_1, x_2)$ be the intrinsic distance between $x_1, x_2 \in S_r$ with respect to the metric γ_r. By the definition of AH manifold, it is easy to see that

$$l(x_1, x_2) = \frac{1}{\sinh r} d_{\mathbb{S}^2}(x_1, x_2)\left(1 + O(r^3)\right). \tag{2.12}$$

On the other hand, let v_1, v_2 be the points of intersections of $\partial B_o(R_2)$ with the geodesics from o to $X^{(r)}(x_1)$ and $X^{(r)}(x_2)$ respectively. Since $X^{(r)}$ is an isometric embedding, the intrinsic distance between $X^{(r)}(x_1)$ and $X^{(r)}(x_2)$ in Σ_r is equal to $l(x_1, x_2)$. Since Σ_r is strictly convex in $\mathbb{H}^3$ by (2.11) and $R_i = \sigma + O(r^3)$, we have

$$l(x_1, x_2) \le d_{\partial B_o(R_2)}(v_1, v_2) + O(r^3)$$

because $l(x_1, x_2)$ is the minimum of lengths of curves in $\mathbb{H}^3$ outside Σ_r which join $X^{(r)}(x_1)$ and $X^{(r)}(x_2)$. Here $d_{\partial B_o(R_2)}$ is the intrinsic distance function on $\partial B_o(R_2)$. So we have

$$l(x_1, x_2) \le \sinh\sigma d_{\mathbb{S}^2}\left(y^{(r)}(x_1), y^{(r)}(x_2)\right) + O(r^2).$$

Using the fact that $\partial B_o(R_1)$ is also strictly convex, one can prove similarly,

$$l(x_1, x_2) \ge \sinh\sigma d_{\mathbb{S}^2}\left(y^{(r)}(x_1), y^{(r)}(x_2)\right) + O(r^2).$$

Combining these two inequalities we have:

$$l(x_1, x_2) = \sinh\sigma d_{\mathbb{S}^2}\left(y^{(r)}(x_1), y^{(r)}(x_2)\right) + O(r^2). \tag{2.13}$$

By (2.12), (2.13) and the fact that $\sinh\sigma = \frac{1}{\sinh r}$, the result follows. □

Let $X^{(r)}$ be the isometric embeddings normalized as above.

Lemma 2.18. *With the above notations, by composing $X^{(r)}$ with isometries of $\mathbb{H}^3$ fixing o, and the resulting isometric embeddings still denoted by $X^{(r)}$, we have:*

$$\lim_{r\to 0} y^{(r)}(x) = x, \quad x \in \mathbb{S}^2.$$

The convergence is uniform in x.

Proof. $x \in \mathbb{S}^2$ is of the form $x = (x^1, x^2, x^3)$ with $\sum_i (x^i)^2 = 1$. Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. By composing with isometries of $\mathbb{H}^3$ fixing o, we may arrange that for all r

$$y^{(r)}(e_1) = e_1, y^{(r)}(e_2) \in \{x^3 = 0, x^2 \geq 0\}, y^{(r)}(e_3) \in \{x^3 \geq 0\}. \tag{2.14}$$

By Lemma 2.17,

$$d_{\mathbb{S}^2}(y^r(e_2), e_1) = d_{\mathbb{S}^2}(y^r(e_2), y^{(r)}(e_1)) = d_{\mathbb{S}^2}(e_2, e_1) + O(r^3).$$

By (2.14), we can conclude that $\lim_{r\to 0} y^{(r)}(e_2) = e_2$. For any $r_n \to 0$ such that $y^{(r_n)}(e_3) \to a = (a^1, a^2, a^3)$ with $a^3 \geq 0$. Then by Lemma 2.17 again, we have

$$d_{\mathbb{S}^2}(e_1, a) = d_{\mathbb{S}^2}(e_2, a) = \frac{\pi}{2}.$$

Hence $a = e_3$. This implies that $\lim_{r\to 0} y^{(r)}(e_3) = e_3$. That is, we have

$$\lim_{r\to\infty} y^{(r)}(e_i) = e_i, \quad \text{for } 1 \leq i \leq 3. \tag{2.15}$$

Now for any $x \in \mathbb{S}^2$ and $r_n \to 0$ such that $\lim_{n\to\infty} y^{(r_n)}(x) = b$. Then by (2.15) and Lemma 2.17, we have

$$d_{\mathbb{S}^2}(e_i, b) = d_{\mathbb{S}^2}(e_i, x), \quad \text{for } 1 \leq i \leq 3.$$

Hence $b = x$ and so $\lim_{r\to 0} y^{(r)}(x) = x$ for all $x \in \mathbb{S}^2$.

We claim that the convergence is uniform. Fix $x_0 \in \mathbb{S}^2$ for any $\epsilon > 0$, by Lemma 2.17, let C be the constant in the lemma, for any $x \in \mathbb{S}^2$ with $d_{\mathbb{S}^2}(x, x_0) < \epsilon$, we have

$$\begin{aligned} d_{\mathbb{S}^2}(y^{(r)}(x), x) \leq & d_{\mathbb{S}^2}(y^{(r)}(x), y^{(r)}(x_0)) + d_{\mathbb{S}^2}(y^{(r)}(x_0), x_0) + d_{\mathbb{S}^2}(x_0, x) \\ \leq & 2 d_{\mathbb{S}^2}(x_0, x) + d_{\mathbb{S}^2}(y^{(r)}(x_0), x_0) + C r^3 \\ \leq & 3\epsilon \end{aligned}$$

provided r is small enough depending only on x_0 and ϵ. Since $\mathbb{S}^2$ is compact, this proves the claim that the convergence is uniform. $\square$

2.2.4 Proof of Theorem 2.3

We now prove our main results. First, we embed $\mathbb{H}^3$ in the $\mathbb{R}^{3,1}$ so that $\mathbb{H}^3 = \{(x^0, x^1, x^2, x^3) \in \mathbb{R}^{3,1} : (x^0)^2 - \sum_{i=1}^{3}(x^i)^2 = 1, x^0 > 0\}$ and the fixed point o in Section 2.2.3 is mapped to the point $(1, 0, 0, 0)$.

Proof of Theorem 2.3. For r small, let $X^{(r)}$ be the embedding of (S_r, γ_r) in $\mathbb{H}^3$ given by Lemma 2.18. With the notations as in section 2.2.3, when considered as an embedding of (S_r, γ_r) in $\mathbb{R}^{3,1}$, $X^{(r)}$ is of the form

$$X^{(r)}(x) = (\cosh\sigma^{(r)}(x), \sinh\sigma^{(r)}(x)\ y^{(r)}(x)). \tag{2.16}$$

Now by Corollary 2.11, Lemmas 2.16 and 2.18, we have as $r \to 0$,

$$\begin{cases} H_0 - H & = \frac{r^3}{2}\mathrm{tr}_{g_0}h + O(r^4), \\ \cosh\sigma^{(r)}(x) & = \coth r + O(r^2) = \frac{1}{r} + o(1), \\ \sinh\sigma^{(r)}(x) & = \frac{1}{\sinh r} + O(r^2) = \frac{1}{r} + o(1), \\ y^{(r)}(x) & = x + o(1). \end{cases} \tag{2.17}$$

As before, $O(r^k)$ represents a quantity with absolute value bounded by Cr^k with C being independent of r and x. Moreover, by Definition 2.1, the volume form

$$d\mu_{\gamma_r} = \left(\frac{1}{\sinh^2 r} + O(r^3)\right) d\mu_{g_0} = (\frac{1}{r^2} + o(1))d\mu_{g_0} \tag{2.18}$$

as $r \to 0$, where $d\mu_{g_0}$ is the volume form of the standard metric g_0. By (2.17) and (2.18), we have

$$
\begin{aligned}
&\int_{S_r}(H_0-H)X^{(r)}d\mu_{\gamma_r}\\
=&\int_{S_r}(H_0-H)\left(\cosh\sigma^{(r)},\sinh\sigma^{(r)}\ y^{(r)}\right)d\mu_{\gamma_r}\\
=&\int_{\mathbb{S}^2}\left((\frac{r^3}{2}\mathrm{tr}_{g_0}h+O(r^4))\left(\frac{1}{r}+o(1),\frac{x}{r}+o(\frac{1}{r})\right)(\frac{1}{r^2}+o(1))\right)d\mu_{g_0}\\
=&\frac{1}{2}\left(\int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)d\mu_{g_0},\int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)xd\mu_{g_0}\right)+o(1).
\end{aligned}
$$

From this the theorem follows. □

Proof of Corollary 2.5. Under the assumptions of the corollary, suppose (M,g) is not isometric to $\mathbb{H}^3$, then by [40, Theorem 2.5], or Theorem 2.2,

$$
\int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)d\mu_{g_0}>\left|\int_{\mathbb{S}^2}\mathrm{tr}_{g_0}(h)xd\mu_{g_0}\right|.
$$

Let $X^{(r)}$ be the isometric embedding of (S_r,γ_r) as in Theorem 2.3, then by the theorem there exists $\epsilon>0$ such that if r is small enough then for any future null vector $\eta=(1,\xi)$,

$$
\left|\int_{S_r}(H_0-H)\langle X^{(r)},\eta\rangle_{\mathbb{R}^{3,1}}d\mu_{\gamma_r}\right|_{\mathbb{R}^{3,1}}\le-\epsilon.
$$

Hence $\int_{S_r}(H_0-H)X^{(r)}d\mu_{\gamma_r}$ is timelike and is future directed. From this and Remark 2.4, it is easy to see that the corollary is true. □

Chapter 3

Positivity of quasi-local mass

The positive mass theorem states that for an asymptotically flat manifold (M, g) such that g behaves like Euclidean at infinity near each end and suppose its scalar curvature is non-negative, then its ADM mass of each end is non-negative, moreover if the ADM mass of one of the end is zero, then (M, g) is actually a Euclidean space. Schoen and Yau [31, 32] proved the positive mass theorem. Witten [41] (see also [26, 3]) gave a simplified proof the positive mass theorem using the spinor method. Since then the method of spinor has been adopted by many people to prove positive mass type theorems or some rigidity results, see for example [34, 1, 23, 38].

In particular, let us look at some results in this direction. M. T. Wang and Yau [38] developed a quasi-local mass for a three dimensional manifold with boundary whose scalar curvature is bounded from below by some negative constant. Using spinor method, they were able to prove that this mass is non-negative. Later on, Shi and Tam [35] also proved a similar result in the three dimensional case, but with a simpler definition of the mass. In this chapter, we will show that the results of Wang-Yau and Shi-Tam also hold in higher dimensions.

In [35], Shi and Tam proved the following:

Theorem 3.1. *([35] Theorem 3.1) Let* (Ω, g) *be a compact* 3*-dimensional ori-*

entable manifold with smooth boundary $\Sigma = \partial\Omega$, homeomorphic to a 2-sphere. Assuming the following conditions:

1. *The scalar curvature R of (Ω, g) satisfies $R > -6k^2$ for some $k > 0$,*

2. *Σ is a topological sphere with Gaussian curvature $K > -k^2$ and mean curvature $H > 0$, so that Σ can be isometrically embedded into $\mathbb{H}^3_{-k^2}$ with mean curvature H_0.*

Then there is a future directed time-like vector-valued function W on Σ such that the vector

$$\int_\Sigma (H_0 - H) W d\Sigma$$

is time-like. Here $W = (x_1, x_2, x_3, \alpha t)$ for some $\alpha > 1$ depending only on the intrinsic geometry of Σ, with $X = (x_1, x_2, x_3, t) \in \mathbb{H}^3_{-k^2} \subset \mathbb{R}^{3,1}$.

In this chapter, we will prove the analogous result in higher dimension for spin manifolds (note that three dimensional orientable manifolds are spin). There are two ingredients which are most important in establishing the main result (Theorem 3.16), one is a monotonicity formula (Lemma 3.6) for the mass expression, the other is a positive mass type theorem (Theorem 3.7). This theorem was originally proved by M.T. Wang and Yau [38] in the three dimensional case. Here we will give a proof in general dimension. In particular, the existence of the Killing spinor fields play an important role in the proof. What is new in the proof of the theorem in higher dimension are two identities involving Killing spinors on the hyperbolic space (Proposition 3.10, 3.9).

This chapter is organized as follows. In Section 3.1, we will first state and prove some preliminary results. In Section 3.2, we will give the proof of a positive mass theorem in general dimension. In Section 3.3, we will then give our main result.

3.1 Preliminaries

In this section, we will state and prove some preliminary results. The setup is as follows.

Let (Ω, g) be a compact n-dimensional manifold with smooth boundary $\Sigma = \partial\Omega$, homeomorphic to a $(n-1)$-sphere. Suppose the scalar curvature R of Ω satisfies $R \geq -n(n-1)k^2$ for some $k > 0$. Let H be the mean curvature of Σ with respect to the outward normal. We assume H is positive, the sectional curvature of Σ is greater than $-k^2$ and Σ can be isometrically embedded uniquely into $\mathbb{H}^n_{-k^2}$, the hyperbolic space of constant sectional curvature $-k^2$. We use the following hyperboloid model for $\mathbb{H}^n_{-k^2}$:

$$\mathbb{H}^n_{-k^2} = \left\{ (x_1, \cdots, x_n, t) \in \mathbb{R}^{n,1} \mid \sum_{i=1}^{n} x_i^2 - t^2 = -\frac{1}{k^2}, t > 0 \right\} \tag{3.1}$$

where $\mathbb{R}^{n,1}$ is the Minkowski space with Lorentz metric $\sum_{i=1}^{n} dx_i^2 - dt^2$. The position vector of $\mathbb{H}^n_{-k^2}$ in $\mathbb{R}^{n,1}$ can be parametrized by

$$X = (x_1, \cdots, x_n, t) = \frac{1}{k}(\sinh(kr)Y, \cosh kr) \tag{3.2}$$

where $Y \in \mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$. Note that r is the geodesic distance of a point from $o = (0, \cdots, 0, 1/k) \in \mathbb{H}^n_{-k^2}$. Without loss of generality we can assume that Σ_0, the embedded image of Σ, encloses a region Ω_0 which contains o.

Let Σ_ρ be the level surface outside Σ_0 in $\mathbb{H}^n_{-k^2}$ with distance ρ from Σ_0. Suppose $F : \Sigma \to \mathbb{H}^n_{-k^2}$ is the embedding with unit outward normal N, then Σ_ρ as a subset of $\mathbb{R}^{n,1}$ is given by ([35] Equation (2.2))

$$X(p, \rho) = \cosh(k\rho)X(p, 0) + \frac{1}{k}\sinh(k\rho)N(p, 0). \tag{3.3}$$

Here for simplicity, (p, ρ) denotes a point Σ_ρ which lies on the geodesic perpendicular to Σ_0 starting from the point $p \in \Sigma_0$ and $X(p, 0) = X(F(p))$.

On $\mathbb{H}^n_{-k^2} \setminus \Omega_0$, the hyperbolic metric can be written as

$$g' = d\rho^2 + g_\rho, \tag{3.4}$$

where g_ρ is the induced metric on Σ_ρ. As in [38], we can perturb the metric to form a new metric on $\mathbb{H}^n_{-k^2} \setminus \Omega_0$

$$g'' = u^2 d\rho^2 + g_\rho \tag{3.5}$$

(note that the induced metrics from g' and g'' on Σ_ρ are the same) with prescribed scalar curvature $-n(n-1)k^2$, where u satisfies ([38] Equation 2.10):

$$\begin{cases} 2H_0 \dfrac{\partial u}{\partial \rho} = 2u^2 \Delta_\rho u + (u - u^3)(R^\rho + n(n-1)k^2), \\ u(p, 0) = \dfrac{H_0(p, 0)}{H(p)}. \end{cases} \tag{3.6}$$

Here Δ_ρ is the Laplacian on Σ_ρ, R^ρ is the scalar curvature of Σ_ρ, $H_0(p, \rho)$ is the mean curvature of Σ_ρ in $(\mathbb{H}^n_{-k^2}, g')$ and $H(p)$ is the mean curvature of $\partial\Omega$ in (Ω, g). The mean curvature of Σ_ρ with respect to the new metric g'' is then

$$H(p, \rho) = \frac{H_0(p, \rho)}{u(p, \rho)}. \tag{3.7}$$

We have the following estimates:

Lemma 3.2 (cf. [38] p. 255-257)**.** *1. For all ρ, $e^{-2k\rho} g_\rho$ is uniformly equivalent to the standard metric on $\mathbb{S}^{n-1}$. Indeed, we can choose a coordinates around any $p \in \Sigma$ such that $g_{ab}(p, \rho) = f\delta_{ab}$, where $f = \sinh^2(k(\mu_a + \rho))/\sinh^2(k\mu_a), e^{2k\rho}$ or $\cosh^2(k(\mu_a + \rho))/\cosh^2(k\mu_a)$ and $\lambda_a(p, 0) = k\coth(k\mu_a), k$ or $k\tanh(k\mu_a)$ is the initial principal curvature.*

2. Let $d\Sigma_\rho$ denotes the volume element of Σ_ρ, then $e^{-(n-1)k\rho} d\Sigma_\rho$ is uniformly equivalent to the volume element $d\mathbb{S}^{n-1}$ of $\mathbb{S}^{n-1}$.

3. The principal curvatures of Σ_ρ with respect to g' is of order $\lambda_a(p, \rho) = k(1 + O(e^{-2k\rho}))$, and therefore $H_0 = (n-1)k + O(e^{-2k\rho})$.

4. $|u - 1| \le Ce^{-nk\rho}$ for some $C > 0$ independent of ρ.

We also have the following long time existence result:

Proposition 3.3 (cf. [38] Theorem 2.1). *1. The solution u of* (3.6) *exists for all time and $v = \lim_{\rho\to\infty} e^{nk\rho}(u-1)$ exists as a smooth function on Σ.*

2. $g'' = u^2 d\rho^2 + g_\rho$ is asymptotically hyperbolic [1] on $M = \mathbb{H}^n_{-k^2} \setminus \Omega_0$ with scalar curvature $-n(n-1)k^2$.

3. Let $A : (TM, g') \to (TM, g'')$ be the Gauge transformation defined by $A\frac{\partial}{\partial\rho} = \frac{1}{u}\frac{\partial}{\partial\rho}$ and $AV = V$ for any vector $V \in T\Sigma_\rho$, then $|A - Id|_{g'} = O(e^{-nk\rho})$ and $|\nabla' A|_{g'} = O(e^{-nk\rho})$.

Since the proofs of the above two results are exactly the same as in [38] except some minor modification, we omit them here.

Lemma 3.4. *(cf. [35] Lemma 3.4) On $\mathbb{H}^n_{-k^2} \setminus \Omega_0$,*

$$H_0 \frac{\partial X}{\partial \rho} + \Delta_\rho X - (n-1)k^2 X = 0.$$

Proof. Under the representation in (3.2) in $\mathbb{H}^n_{-k^2}$, the Laplacian in $\mathbb{H}^n_{-k^2}$ is given by

$$\Delta_{\mathbb{H}^n_{-k^2}} = \frac{\partial^2}{\partial\rho^2} + (n-1)k \coth kr \frac{\partial}{\partial\rho} + k^2 \sinh^{-2} kr \Delta_{\mathbb{S}^{n-1}}.$$

By $\Delta_{\mathbb{S}^{n-1}} Y = -(n-1)Y$ for $Y \in \mathbb{S}^{n-1}$ and (3.2), $\Delta_{\mathbb{H}^n_{-k^2}} X = nk^2 X$.

On the other hand, under the foliation by Σ_ρ, the $\Delta_{\mathbb{H}^n_{-k^2}}$ is given by

$$\Delta_{\mathbb{H}^n_{-k^2}} = \frac{\partial^2}{\partial\rho^2} + H_0 \frac{\partial}{\partial\rho} + \Delta_\rho$$

where Δ_ρ is the Laplacian on Σ_ρ. So using (3.3),

$$nk^2 X = \frac{\partial^2}{\partial\rho^2} X + H_0 \frac{\partial}{\partial\rho} X + \Delta_\rho X = k^2 X + H_0 \frac{\partial}{\partial\rho} X + \Delta_\rho X.$$

□

Let $B_0(R_1)$ and $B_0(R_2)$ be geodesic balls in $\mathbb{H}^n_{-k^2}$ such that $B_0(R_1) \subset D \subset B_0(R_2)$. We define $W = (x_1, x_2, \cdots, x_n, \alpha t)$ with

$$\alpha = \coth kR_1 + \frac{1}{\sinh kR_1}\left(\frac{\sinh^2 kR_2}{\sinh^2 kR_1} - 1\right)^{\frac{1}{2}},$$

where $X=(x_1,x_2,\cdots,x_n,t)$ is the position vector of Σ_ρ in $\mathbb{R}^{n,1}$.

By the same argument we have

Lemma 3.5. *On $\mathbb{H}^n_{-k^2}\setminus\Omega_0$, $H_0\frac{\partial W}{\partial\rho}+\Delta_\rho W-(n-1)k^2W=0$.*

Lemma 3.6. *(cf. [35] Equation 3.8) On $\mathbb{H}^n_{-k^2}\setminus\Omega_0$,*

$$\begin{aligned}&\frac{d}{d\rho}\left(\int_{\Sigma_\rho}(H_0-H)X\,d\Sigma_\rho\right)\\ =&-\int_{\Sigma_\rho}u^{-1}(u-1)^2\left(\left(R^\rho+(n-1)(n-2)k^2\right)\frac{X}{2}+H_0\frac{\partial X}{\partial\rho}\right)d\Sigma_\rho.\end{aligned}$$

Proof. By (3.6) and the divergence theorem,

$$\begin{aligned}&\frac{d}{d\rho}\left(\int_{\Sigma_\rho}(H_0-H)X\,d\Sigma_\rho\right)\\ =&\frac{d}{d\rho}\left(\int_{\Sigma_\rho}H_0(1-u^{-1})X\,d\Sigma_\rho\right)\\ =&\int_{\Sigma_\rho}\Big(\frac{\partial H_0}{\partial\rho}(1-u^{-1})X+H_0u^{-2}\frac{\partial u}{\partial\rho}X+H_0(1-u^{-1})\frac{\partial X}{\partial\rho}\\ &+H_0^2(1-u^{-1})X\Big)d\Sigma_\rho\\ =&\int_{\Sigma_\rho}\Big(\big(\frac{\partial H_0}{\partial\rho}+H_0^2\big)(1-u^{-1})X \qquad (3.8)\\ &+\left(\Delta_\rho u+\frac{1}{2}(u^{-1}-u)(R^\rho+n(n-1)k^2)\right)X+H_0(1-u^{-1})\frac{\partial X}{\partial\rho}\Big)d\Sigma_\rho\\ =&\int_{\Sigma_\rho}\Big(\big(\frac{\partial H_0}{\partial\rho}+H_0^2\big)(1-u^{-1})X+\frac{1}{2}(u^{-1}-u)(R^\rho+n(n-1)k^2)X\\ &+H_0(1-u^{-1})\frac{\partial X}{\partial\rho}+(u-1)\Delta_\rho X\Big)d\Sigma_\rho\\ =&\int_{\Sigma_\rho}(I+II+III+IV)d\Sigma_\rho\end{aligned}$$

where we have used (3.6) in line 4 and divergence theorem in line 5. The Gauss equation gives

$$R^\rho=-(n-1)(n-2)k^2+H_0^2-|A|^2 \tag{3.9}$$

where A is the second fundamental form of Σ_ρ with respect to the hyperbolic metric g'. By the evolution equation of H_0 ([38] Equation 2.4) and the Gauss equation (3.9),

$$\frac{\partial H_0}{\partial \rho} = -|A|^2 + (n-1)k^2 = R^\rho + (n-1)^2k^2 - H_0^2.$$

So $I = (R^\rho + (n-1)^2k^2)(1-u^{-1})X$.

Direct calculation gives

$$\begin{aligned} &(R^\rho + (n-1)^2k^2)(1-u^{-1}) + \frac{1}{2}(u^{-1}-u)(R^\rho + n(n-1)k^2) \\ =& -\frac{1}{2}u^{-1}(u-1)^2(R^\rho + (n-1)(n-2)k^2) - (n-1)(u-1)k^2. \end{aligned}$$

So we have

$$I + II = (-\frac{1}{2}u^{-1}(u-1)^2\left(R^\rho + (n-1)(n-2)k^2\right) - (n-1)(u-1)k^2)X.$$

By lemma 3.4, $\Delta_\rho X - (n-1)k^2X = -H_0\frac{\partial X}{\partial \rho}$. Therefore

$$\begin{aligned} &I + II + III + IV \\ =& -\frac{1}{2}u^{-1}(u-1)^2\left(R^\rho + (n-1)(n-2)k^2\right)X \\ &+ (u-1)\left(\frac{H_0}{u}\frac{\partial X}{\partial \rho} + \Delta_\rho X - (n-1)k^2X\right) \\ =& -\frac{1}{2}u^{-1}(u-1)^2\left(R^\rho + (n-1)(n-2)k^2\right)X + (u-1)(u^{-1}-1)H_0\frac{\partial X}{\partial \rho} \\ =& -u^{-1}(u-1)^2\left(\left(R^\rho + (n-1)(n-2)k^2\right)\frac{X}{2} + H_0\frac{\partial X}{\partial \rho}\right). \end{aligned}$$

This together with (3.8) gives the result. □

3.2 A Positive mass theorem

We will need the following positive mass theorem ([38] Theorem 6.1) which was proved by M.T. Wang and Yau when $n = 3$.

Theorem 3.7 (Wang-Yau)**.** *Let $n \geq 3$ and (Ω, g) is a n-dimensional compact spin manifold with nonempty smooth boundary which is a topological sphere. Suppose the scalar curvature R of Ω satisfies $R \geq -n(n-1)k^2$, the sectional curvature* *of its boundary Σ satisfies $K > -k^2$, the mean curvature of the boundary with respect to outward unit normal is positive, and Σ can be isometrically embedded uniquely into $\mathbb{H}^n_{-k^2}$ in $\mathbb{R}^{n,1}$. Then*

$$\lim_{\rho\to\infty}\int_{\Sigma_\rho}(H_0 - H)X\cdot\zeta \le 0$$

for any future-directed null vector ζ in $\mathbb{R}^{n,1}$, where H_0, H are functions in (p,ρ) as in (3.7)
In other words, $\lim_{\rho\to\infty}\int_{\Sigma_\rho}(H_0 - H)X d\Sigma_\rho$ is a future-directed non-spacelike vector.

As a corollary,

Corollary 3.8. *With the same assumptions as in Theorem 3.7,*

$$\lim_{\rho\to\infty}\int_{\Sigma_\rho}(H_0 - H)\cosh kr d\Sigma_\rho \ge 0$$

where r is defined in (3.2)*.*

3.2.1 Killing spinors on $(\mathbb{H}^n_{-k^2}, g')$

The proof of Theorem 3.7 requires the existence of Killing spinor fields (i.e. a section of the spinor bundle (see for example [21]) $S(\mathbb{H}^n_{-k^2}, g')$ satisfying the Killing equation (3.10)) on the hyperbolic space. A Killing spinor ϕ' on $(\mathbb{H}^n_{-k^2}, g')$ satisfies the equation

$$\nabla'_V\phi' + \frac{\sqrt{-1}}{2}kc'(V)\phi' = 0 \text{ for any tangent vector } V \tag{3.10}$$

where $c'(V)$ is the Clifford multiplication by V and ∇' is the spin connection (with respect to the hyperbolic metric g'). The Killing spinors on hyperbolic spaces were studied by Baum [5]. Baum proved that on $\mathbb{H}^n_{-k^2}$, the set of all Killing spinors is parametrized by $a \in \mathbb{C}^{2^m}$, $m = \lfloor \frac{n}{2} \rfloor$ (integer part of $\frac{n}{2}$). We need the following two propositions.

Proposition 3.9. *Let $\phi'_{a,0}$ be the Killing spinor on $(\mathbb{H}^n_{-k^2}, g')$, corresponding to $a \in \mathbb{C}^{2^m}$, $m = \lfloor \frac{n}{2} \rfloor$, then*

$$|\phi'_{a,0}|^2_{g'} = -2kX \cdot \zeta_a$$

where $\cdot$ denotes the Lorentz inner product in $\mathbb{R}^{n,1}$ and

$$\zeta_a = \sum_{j=1}^{n} \langle \sqrt{-1} c(e_j) a, a \rangle e_j - \langle \sqrt{-1} c(e_0) a, a \rangle e_0. \tag{3.11}$$

Here $\langle , \rangle$ is the inner product in $\mathbb{C}^{2^m}$, $c(e_j)$ denotes the Clifford multiplication by the Clifford matrices (as defined in [5]) for the orthonormal basis $\frac{\partial}{\partial t} = e_0$, $\frac{\partial}{\partial x_j} = e_j$ in $\mathbb{R}^{n,1}$ $(1 \le j \le n)$ and $c(e_0)$ is defined to be $\begin{pmatrix} \sqrt{-1} & & \\ & \ddots & \\ & & \sqrt{-1} \end{pmatrix}$.

Proof. Let $k = 1$ for simplicity. Baum ([5, Theorem 1], $\mu = -\frac{k}{2}$) proved that in the ball model for $\mathbb{H}^n$, the Killing spinor can be expressed as (note that the spinor bundle is trivial)

$$\phi = \phi'_{a,0}(x) = \sqrt{\frac{2}{1 - |x|^2}} (a - \sqrt{-1} c(x) a)$$

where $c(x)a = \sum_{j=1}^{n} x_j c(e_j) a$ for $x = (x_1, \cdots, x_n)$. It is easily computed that

$$|\phi|^2 = 2 \left(\frac{1 + |x|^2}{1 - |x|^2} |a|^2 - \frac{2}{1 - |x|^2} \langle \sqrt{-1} c(x) a, a \rangle \right).$$

The change of coordinates from the ball model to the hyperboloid model is given by

$$X = \left(\frac{2x}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2}\right) \in \mathbb{R}^{n,1}.$$

So

$$\begin{aligned}
-2X \cdot \zeta_a &= -4 \sum_{j=1}^{n} \frac{x_j}{1 - |x|^2} \langle \sqrt{-1} c(e_j) a, a \rangle + 2 \frac{1 + |x|^2}{1 - |x|^2} |a|^2 \\
&= 2 \left(\frac{1 + |x|^2}{1 - |x|^2} |a|^2 - \frac{2}{1 - |x|^2} \langle \sqrt{-1} c(x) a, a \rangle \right) \\
&= |\phi|^2.
\end{aligned}$$

□

Proposition 3.10. *For every null vector $\zeta \in \mathbb{R}^{n,1}$ ($n \geq 2$), $\zeta = \zeta_a$ for some $a \in \mathbb{C}^{2^m}$, where $m = \lfloor \frac{n}{2} \rfloor$ and ζ_a is defined in* (3.11).

Proof. Define $\eta_a = \sum_{j=1}^{n} \langle \sqrt{-1} c(e_j) a, a \rangle e_j$. As $\zeta_a = \sum_{j=1}^{n} \langle \sqrt{-1} c(e_j) a, a \rangle e_j + |a|^2 e_0$, it suffices to prove that for any $X \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$, there exists $a \in \mathbb{C}^{2^m}$ with $|a| = 1$ such that $\eta_a = X$. We divide into two cases: (i) n is odd and (ii) n is even.

(i) For the odd case where $n = 2m+1$, we apply induction on m. When $2m+1 = 3$, this is done in [38] (p.17). We state it here for later use. The three Clifford matrices for $n = 3$ are g_1, g_2 and $\sqrt{-1}T$ (see [5, p. 206]), where

$$g_1 = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, g_2 = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}.$$

So for any $\vec{z} \in \mathbb{S}^2$, there exists $a \in \mathbb{C}^2$ with $|a| = 1$ such that

$$\vec{z} = (\langle \sqrt{-1} g_1(a), a \rangle, \langle \sqrt{-1} g_2(a), a \rangle, \langle -T(a), a \rangle). \tag{3.12}$$

Assume the result is true for $n = 2m-1$, and denote the Clifford matrices in dimension $2m-1$ simply by $\{c_j\}_{j=1}^{2m-1}$. Let $\{d_j\}_{j=1}^{2m+1}$ be the Clifford matrices in dimension $2m+1$, as defined in [5, p. 206 Equation (2)]. Then it is easily seen that

$$\begin{cases} d_j = I \otimes c_j \quad \text{for } j = 1, \cdots, 2m-2,\ I \text{ is the } 2 \times 2 \text{ identity matrix}, \\ d_{2m-1} = -\sqrt{-1} g_1 \otimes c_{2m-1}, \\ d_{2m} = -\sqrt{-1} g_2 \otimes c_{2m-1}, \\ d_{2m+1} = T \otimes c_{2m-1}. \end{cases} \tag{3.13}$$

Now let $X \in \mathbb{S}^{2m}$, then $X = (y_1, y_2, \cdots, y_{2m-1} \vec{z})$ for some $y = (y_1, \cdots, y_{2m-1}) \in \mathbb{S}^{2m-2}$ and $\vec{z} \in \mathbb{S}^2$. By induction assumption, there exists $b \in \mathbb{C}^{2^{m-1}}$ with $|b| = 1$ such that

$$y = (\langle \sqrt{-1} c_1(b), b \rangle, \cdots, \langle \sqrt{-1} c_{2m-1}(b), b \rangle)$$

and by (3.12), there exists $a \in \mathbb{C}^2$ with $|a| = 1$ such that

$$-\vec{z} = (\langle \sqrt{-1} g_1(a), a \rangle, \langle \sqrt{-1} g_2(a), a \rangle, \langle -T(a), a \rangle).$$

Combining these with (3.13), it is easily seen that $\eta_{a\otimes b} = X$.

(ii) For the even case, we also apply induction on m. When $n = 2$, the two Clifford matrices are g_1 and g_2 and $\eta_a = (-|a_1|^2 + |a_2|^2, -a_1\overline{a}_2 - a_2\overline{a}_1)$ where $a = (a_1, a_2) \in \mathbb{C}^2$. For $X = (\cos\theta, \sin\theta) \in \mathbb{S}^1$, just take $a = (-\sin\frac{\theta}{2}, \cos\frac{\theta}{2})$ so that $\eta_a = X$.

Assume the result is true for $n = 2m$ and denote $\{c_j\}_{j=1}^{2m}$ to be the corresponding Clifford matrices as defined in [5, p. 206 Equation (1)]. Let $\{d_j\}_{j=1}^{2m+2}$ be the Clifford matrices for $n = 2m + 2$. Then it is easily seen that

$$\begin{cases} d_1 = I \otimes g_1 \quad \text{where } I \text{ is the } 2^m \times 2^m \text{ identity matrix,} \\ d_2 = I \otimes g_2 \quad \text{where } I \text{ is the } 2^m \times 2^m \text{ identity matrix,} \\ d_{j+2} = c_j \otimes T \quad \text{for } j = 1, \cdots, 2m. \end{cases} \tag{3.14}$$

Now let $X \in \mathbb{S}^{2m+1}$, then $X = (z_1, z_2, z_3\vec{y})$ for some $(z_1, z_2, z_3) \in \mathbb{S}^2$ and $\vec{y} \in \mathbb{S}^{2m-1}$. By (3.12), there exists $b \in \mathbb{C}^2$ with $|b| = 1$ such that

$$(z_1, z_2, -z_3) = (\langle\sqrt{-1}g_1(b), b\rangle, \langle\sqrt{-1}g_2(b), b\rangle, \langle -T(b), b\rangle)$$

and by induction assumption, there exists $a \in \mathbb{C}^{2^{m-1}}$ with $|a| = 1$ such that

$$\vec{y} = (\langle\sqrt{-1}c_1(a), a\rangle, \cdots, \langle\sqrt{-1}c_{2m}(a), a\rangle).$$

Combining these with (3.14), it is easily seen that $\eta_{a\otimes b} = X$. □

3.2.2 The hypersurface Dirac operator

In this subsection, we will give some general results for the hypersurface Dirac operator. Most of the materials in this section can be found, for example, in [16].

Recall that on a spinor bundle $S(M^n)$ over (M, g), the Dirac operator D is defined to be

$$D\psi = \sum_{i=1}^{n} c_M(e_i)\nabla^M_{e_i}\psi$$

for any spinor $\psi \in \Gamma(S(M))$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame on M, c_M is the Clifford multiplication and ∇^M is the spin connection on $S(M)$. The local formula for ∇^M is given by [21, Theorem 4.14]

$$\nabla^M_{e_i}\psi = e_i(\psi) + \frac{1}{2}\sum_{j<k}^{n}\langle \nabla_{e_i}e_j, e_k\rangle c_M(e_j)c_M(e_k)\psi,$$

where $\{e_i\}_{i=1}^n$ are orthonormal frames on M. For simplicity, let us write c for c_M and ∇ for ∇^M.

Now, for a spin manifold M, if $\Sigma \subset M$ is an oriented smooth hypersurface, then M induces a natural spin structure on Σ, compatible with the induced orientation from M.

We let $S := S(M^n)|_\Sigma$, the restriction of the spinor bundle of M to Σ. Then it can be shown that $S = S(\Sigma)$ when n is odd and $S = S(\Sigma) \oplus S(\Sigma)$ when n is even. We will work on S instead of $S(\Sigma)$.

Definition 3.11. *We define the hypersurface spin connection ∇^S, the hypersurface Clifford multiplication c_S and the hypersurface Dirac operator D^S on S by*

$$\begin{aligned}
\nabla^S_X\psi &= \nabla_X\psi + \frac{1}{2}c(\nu)c(B(X))\psi, \\
c_S(X) &= -c(\nu)c(X), \\
D^S\psi &= \sum_{a=1}^{n-1} c_S(e_a)\nabla^S_{e_a}\psi.
\end{aligned}$$

where ν is a fixed unit normal (outward if this makes sense) and B is the shape operator on Σ, i.e. $B(X) = -\nabla_X\nu$.

In local formula, for $\{e_a\}_{a=1}^{n-1}$ orthonormal on Σ and $e_n = \nu$ be the unit outward normal,

$$\nabla^S_{e_a}\psi = \nabla_{e_a}\psi + \frac{1}{2}\sum_{b=1}^{n-1} h_{ab}c(e_b)c(e_n)\psi. \tag{3.15}$$

(It can be verified that $\nabla^S = \nabla^\Sigma \oplus \nabla^\Sigma$ and $c_S = c_\Sigma \oplus -c_\Sigma$ when n is even.)

Definition 3.12. *We define the Killing spin connection $\widehat{\nabla}$, Killing Dirac operator $\widehat{D}$ and the Killing boundary operator $\widehat{B}$ respectively by*

$$\begin{aligned}
\widehat{\nabla}_V\psi &= \nabla_V\psi + \frac{\sqrt{-1}}{2}kc(V)\psi,\\
\widehat{D}\psi &= \sum_{i=1}^{n} c(e_i)\widehat{\nabla}_{e_i}\psi,\\
\widehat{B}\psi &= \sum_{a=1}^{n-1} c(e_n)c(e_a)\widehat{\nabla}_{e_a}\psi = \widehat{\nabla}_{e_n}\psi + \widehat{D}\psi.
\end{aligned} \tag{3.16}$$

Actually $\widehat{B}$ is the boundary operator for the Lichnerowicz type formula ([38] Equation 3.2): for any bounded region U with smooth boundary, we have

$$\int_U \left(\langle\widehat{\nabla}\psi, \widehat{\nabla}\varphi\rangle + \frac{1}{4}(R+n(n-1))\langle\psi,\varphi\rangle - \langle\widehat{D}\psi, \widehat{D}\varphi\rangle\right) = \int_{\partial U}\langle\psi, \widehat{B}\varphi\rangle. \tag{3.17}$$

where R is the scalar curvature.

From now on until the end of this section, the indices a, b, c run from 1 to $n-1$ and i, j, k run from 1 to n. Repeated indices will be summed over.

Proposition 3.13. *Let ψ be a spinor on M and H is the mean curvature of $\Sigma \subset M$. Then on Σ,*

$$\widehat{B}\psi = -D^S\psi - \frac{H}{2}\psi - \frac{\sqrt{-1}}{2}k(n-1)c(e_n)\psi.$$

Proof. We have $\widehat{B}\psi = c(e_n)c(e_a)\widehat{\nabla}_{e_a}\psi$. Using (3.15),

$$c(e_n)c(e_a)\widehat{\nabla}_{e_a}\psi = c(e_n)c(e_a)\nabla^S_{e_a}\psi + \frac{1}{2}h_{ab}c(e_a)c(e_b) - \frac{\sqrt{-1}}{2}k(n-1)c(e_n)\psi.$$

We have $c_S(e_a) = c(e_a)c(e_n)$, so $c(e_n)c(e_a)\nabla^S_{e_a} = -D^S$. Also, $h_{ab}c(e_a)c(e_b) = -H$. The result follows. □

Let us now return to the hyperbolic space. More precisely, define $M = \mathbb{H}^n_{-k^2} \setminus \Omega_0$. Let $A : (TM, g') \to (TM, g'')$ be the Gauge transformation defined by $A\frac{\partial}{\partial\rho} = \frac{1}{u}\frac{\partial}{\partial\rho}$ (u as defined in (3.6)) and $AV = V$ for any vector V tangential to Σ_ρ. A can be lifted to the spinor bundles as an isometry [1], i.e. $A : S(M, g') \to S(M, g'')$. Also,

$$A(c'(X)\psi) = c''(AX)A\psi$$

where c' (resp. c'') denotes the Clifford multiplication associated to g' (resp. g''). We will also denote by e''_n (resp. e'_n) to denote the unit outward normal of Σ_ρ with respect to g'' (resp. g').

Proposition 3.14. *Let ϕ'_0 be a Killing spinor with respect to ∇' on $M = \mathbb{H}^n_{-k^2}\setminus\Omega_0$ and $\phi_0 = A\phi'_0$. Let D^S be the hypersurface Dirac operator on Σ_ρ with respect to (M, g'') as defined in (3.16). Then on Σ_ρ,*

$$-D^S\phi_0 = \frac{H_0}{2}\phi_0 + \frac{\sqrt{-1}}{2}k(n-1)c''(e''_n)\phi_0.$$

(Recall that H_0 is the mean curvature of Σ_ρ with respect to g'.)

Proof. (The proof is the same as in [38] except we have to replace ∇^{Σ_ρ} by ∇^S etc, corresponding to Σ_ρ.)

For $\overline{\nabla}_{e_a}\psi := A\nabla_{e_a}(A^{-1}\psi)$, we have

$$\overline{\nabla}_{e_a}\phi_0 = -\frac{\sqrt{-1}}{2}kc''(e_a)\phi_0 \tag{3.18}$$

Consider

$$\begin{aligned}\overline{\nabla}_{e_a}\psi =& e_a(\psi) + \frac{1}{2}\sum_{b<c}^{n-1} g''(\overline{\nabla}_{e_a}e_b, e_c)c''(e_b)c''(e_c)\psi \\ &+ \frac{1}{2}\sum_{b=1}^{n-1} g''(\overline{\nabla}_{e_a}e_b, e''_n)c''(e_b)c''(e''_n)\psi.\end{aligned} \tag{3.19}$$

Note that $g''(\overline{\nabla}_{e_a}e_b, e_c) = g''(A\nabla'_{e_a}A^{-1}e_b, e_c) = g''(A\nabla'_{e_a}e_b, Ae_c) = g'(\nabla'_{e_a}e_b, e_c) = g''(\nabla''_{e_a}e_b, e_c)$. ($g'$ and g'' induces the same metric on Σ_ρ.) Also, $g''(\overline{\nabla}_{e_a}e_b, e''_n) = g''(A\nabla'_{e_a}A^{-1}e_b, Ae'_n) = g'(\nabla'_{e_a}e_b, e'_n) = -h^0_{ab}$. So (3.19) becomes

$$\begin{aligned}\overline{\nabla}_{e_a}\psi &= e_a(\psi) + \frac{1}{2}\sum_{b<c} g''(\nabla''_{e_a}e_b, e_c)c''(e_b)c''(e_c)\psi - \frac{1}{2}h^0_{ab}c''(e_b)c''(e''_n)\psi \\ &= \nabla^S_{e_a}\psi - \frac{1}{2}h^0_{ab}c''(e_b)c''(e''_n)\psi \quad \text{by (3.15)}.\end{aligned} \tag{3.20}$$

Note that by definition of D^S and c_S,

$$D^S\psi = c_S(e_a)\nabla^S_{e_a}\psi = -c''(e''_n)c''(e_a)\nabla^S_{e_a}\psi.$$

So using (3.20) and (3.18),

$$
\begin{aligned}
D^S\phi_0 &= -c''(e_n'')c''(e_a)\left(\overline{\nabla}_{e_a}\phi_0 + \frac{1}{2}h^0_{ab}c''(e_b)c''(e_n'')\phi_0\right)\\
&= -c''(e_n'')c''(e_a)\left(-\frac{\sqrt{-1}}{2}kc''(e_a)\phi_0 + \frac{1}{2}h^0_{ab}c''(e_b)c''(e_n'')\phi_0\right)\\
&= -\frac{\sqrt{-1}}{2}k(n-1)c''(e_n'')\phi_0 - \frac{H_0}{2}\phi_0.
\end{aligned}
$$

□

Proposition 3.15. *With the assumptions in Theorem 3.7, let $\phi'_{a,0}$ be a Killing spinor with respect to g' and $\phi_0 = A\phi'_0$ on M. Then the limit $\lim_{\rho\to\infty}\int_{\Sigma_\rho}(H_0 - H)|\phi_0|^2_{g''}d\Sigma_\rho$ exists.*

Proof. $\phi'_0 = \phi'_{a,0}$ as in Proposition 3.9. By Proposition 3.9, $|\phi_{a,0}|^2_{g''} = -2kX\cdot\zeta_a$. By (3.3), $e^{-k\rho}X(p,\rho) \to \gamma(p) = X(p,0) + \frac{1}{k}N(p,0)$.

Also $e^{nk\rho}(H_0 - H) = H_0 e^{nk\rho}(1-u^{-1}) \to (n-1)kv$ as given by Lemma 3.2 and Proposition 3.3. By Lemma 3.2 again, $e^{-(n-1)k\rho}d\Sigma_\rho$ tends to a measure $d\mu$ on Σ, induced by the metric $g_\infty = \lim_{\rho\to\infty} e^{-2\rho}g_\rho$.

All the above limits are uniform in ρ. Thus we have

$$
\begin{aligned}
\int_{\Sigma_\rho}(H_0 - H)|\phi_0|^2_{g''}d\Sigma_\rho &= -2k\int_\Sigma H_0(e^{nk\rho}(1-u^{-1}))(e^{-k\rho}X(p,\rho)\cdot\zeta_a)e^{-(n-1)k\rho}d\Sigma_\rho\\
&\to -2(n-1)k^2\int_\Sigma v(\gamma\cdot\zeta_a)d\mu.
\end{aligned}
$$

□

3.2.3 Proof of Theorem 3.7

Following the ideas in [38] Theorem 6.1, we now give the proof of Theorem 3.7.

Proof of Theorem 3.7. Define $g'' = u^2d\rho^2 + g_\rho$ on $M = \mathbb{H}^n_{-k^2}\setminus\Omega_0$ as in (3.5), with u satisfying (3.6). Let $\widetilde{g}$ be the metric defined on $\widetilde{M} = M\cup_F\Omega$ such that $\widetilde{g} = g$ on Ω and $\widetilde{g} = g''$ on M, where F is the embedding of Ω into $\mathbb{H}^n_{-k^2}$. Note that $\tilde{g}$

is Lipschitz near $\partial\Omega$, i.e. there is a smooth coordinates around $\partial\Omega$ such that the coefficients $\widetilde{g}_{ij}$ are Lipschitz.

Let $\widehat{\nabla}_V = \widetilde{\nabla}_V + \frac{\sqrt{-1}}{2}k\widetilde{c}(V)$ and $\widehat{D} = \widetilde{c}(e_i)\widehat{\nabla}_{e_i}$ be the Killing connection and Killing-Dirac operator associated with $\widetilde{g}$ respectively. (All inner products and norms in this proof are taken with respect to $\tilde{g}$ unless otherwise stated.)

Let ϕ_0' be a Killing spinor on $\mathbb{H}^n_{-k^2}$ and $\phi_0 = A\phi_0'$ on M, we claim that there is a (Killing harmonic) spinor ϕ with $\widehat{D}\phi = 0$ on $\widetilde{M}$ such that

$$0 \le \lim_{m\to\infty}\int_{\Sigma_{\rho_m}} \langle \phi, \widehat{B}\phi\rangle = \lim_{m\to\infty}\int_{\Sigma_{\rho_m}} \langle \phi_0, \widehat{B}\phi_0\rangle \tag{3.21}$$

where $\rho_m \to \infty$ and $\widehat{B}$ is the boundary operator with respect to $\tilde{g}$ as in (3.16).

Since we are only interested in the asymptotic behavior, by cutting off, we can assume that ϕ_0 can be extended smoothly on the whole $\widetilde{M}$. Then near infinity (i.e. outside a compact set), for $\overline{\nabla} = A\nabla A^{-1}$, we have

$$\begin{cases} \overline{\nabla}_{e_a}\phi_0 = -\frac{\sqrt{-1}}{2}\widetilde{c}(e_a)\phi_0, \\ \overline{\nabla}_{\frac{\partial}{\partial\rho}}\phi_0 = -\frac{\sqrt{-1}}{2}\frac{1}{u}\widetilde{c}(\frac{\partial}{\partial\rho})\phi_0. \end{cases}$$

So

$$\begin{cases} \widehat{\nabla}_{e_a}\phi_0 = \widetilde{\nabla}_{e_a}\phi_0 + \frac{\sqrt{-1}}{2}\widetilde{c}(e_a)\phi_0 = (\widetilde{\nabla}_{e_a} - \overline{\nabla}_{e_a})\phi_0, \\ \widehat{\nabla}_{\frac{\partial}{\partial\rho}}\phi_0 = \widetilde{\nabla}_{\frac{\partial}{\partial\rho}}\phi_0 + \frac{\sqrt{-1}}{2}\widetilde{c}(\frac{\partial}{\partial\rho})\phi_0 = (\widetilde{\nabla}_{\frac{\partial}{\partial\rho}} - u\overline{\nabla}_{\frac{\partial}{\partial\rho}})\phi_0. \end{cases}$$

By the estimates in Lemma 2.1 of [1], we have

$$|(\widetilde{\nabla} - \overline{\nabla})\psi| \le C|A^{-1}||\nabla' A||\psi|.$$

By Proposition 3.3, $|A^{-1}||\nabla' A| = O(e^{-nk\rho})$. Also $|\phi_0|^2 = O(e^{k\rho})$ by Proposition 3.9, so $|\widehat{\nabla}\phi_0| = O(e^{-(n-\frac{1}{2})k\rho})$. By Lemma 3.2, the volume element of $(\widetilde{M}, \widetilde{g})$ is of order $e^{(n-1)k\rho}$. We then have $\widehat{\nabla}\phi_0$, and therefore $\widehat{D}\phi_0$, are both in $L^2(\widetilde{M}, \widetilde{g})$.

We now find $\phi_1 \in W^{1,2}$ such that $\widehat{D}\phi_1 = \widehat{D}\phi_0$ as follows. We define a linear map on $W^{1,2}$ by

$$l(\psi) = \int_{\widetilde{M}} \langle \widehat{D}\psi, \widehat{D}\phi_0\rangle.$$

Define the sesquilinear form B on $W^{1,2}$ by

$$B(\psi,\varphi)=\int_{\widetilde{M}}\langle\widehat{D}\psi,\widehat{D}\varphi\rangle.$$

We claim that B is bounded and coercive.

Let $\widetilde{M}_\rho$ be the region in $\widetilde{M}$ bounded by Σ_ρ and let $\psi,\varphi\in C_c^\infty$. On $\widetilde{M}_\rho\setminus\Omega$, $R=-n(n-1)$, so by the Lichnerowicz formula (3.17), Proposition 3.13 and the definition of $\widehat{B}$,

$$\int_{\widetilde{M}_\rho\setminus\Omega}(\langle\widehat{\nabla}\psi,\widehat{\nabla}\varphi\rangle-\langle\widehat{D}\psi,\widehat{D}\varphi\rangle)=\int_{\partial\Omega}\langle\psi,(D^S+\frac{H}{2}+\sqrt{-1}c''(\nu))\varphi\rangle+\int_{\Sigma_\rho}\langle\psi,(\nabla_\nu+c''(\nu)\widehat{D})\varphi\rangle.$$

On $\Omega\subset\widetilde{M}_\rho$,

$$\int_\Omega(\langle\widehat{\nabla}\psi,\widehat{\nabla}\varphi\rangle-\langle\widehat{D}\psi,\widehat{D}\varphi\rangle+\frac{1}{4}(R+n(n-1))\langle\psi,\varphi\rangle)=\int_{\partial\Omega}\langle\psi,-(D^S+\frac{H}{2}+\sqrt{-1}c''(\nu))\varphi\rangle.$$

To be precise, H in the two equations above are the mean curvatures of $\partial\Omega$ with respect to g'' and g respectively, but since they agree ((3.6), (3.7)), so adding them up gives

$$\int_{\widetilde{M}_\rho}\left(\langle\widehat{\nabla}\psi,\widehat{\nabla}\varphi\rangle-\langle\widehat{D}\psi,\widehat{D}\varphi\rangle+\frac{1}{4}(R+n(n-1))\langle\psi,\varphi\rangle\right)=\int_{\Sigma_\rho}\langle\psi,(\nabla_\nu+c''(\nu)\widehat{D})\varphi\rangle.$$

As $R=-n(n-1)$ outside Ω, so

$$\begin{aligned}B(\psi,\varphi)=\int_{\widetilde{M}}\langle\widehat{D}\psi,\widehat{D}\varphi\rangle&=\int_{\widetilde{M}}\left(\langle\widehat{\nabla}\psi,\widehat{\nabla}\varphi\rangle+\frac{1}{4}(R+n(n-1))\langle\psi,\varphi\rangle\right)\\&\le C|\psi|_{W^{1,2}}|\varphi|_{W^{1,2}}.\end{aligned}$$

So B is bounded on $W^{1,2}$. On the other hand, as $R\ge -n(n-1)$, for $\psi\in C_c^\infty$,

$$\begin{aligned}\int_{\widetilde{M}}|\widehat{D}\psi|^2\ge\int_{\widetilde{M}}|\widehat{\nabla}\psi|^2&=\int_{\widetilde{M}}\left(|\nabla\psi|^2+\frac{n|\psi|^2}{4}+\frac{\sqrt{-1}}{2}(\langle D\psi,\psi\rangle-\langle\psi,D\psi\rangle)\right)\\&=\int_{\widetilde{M}}\left(|\nabla\psi|^2+\frac{n|\psi|^2}{4}\right)\\&\ge C|\psi|^2_{W^{1,2}}.\end{aligned}$$

So B is also coercive. Then by Lax-Milgram theorem, there exists $\phi_1\in W^{1,2}$ such that $B(\phi_1,\psi)=l(\psi)$ for all $\psi\in W^{1,2}$. i.e.

$$\int_{\widetilde{M}}\langle\widehat{D}(\phi_1-\phi_0),\widehat{\psi}\rangle=0.$$

Let $\phi = \phi_1 - \phi_0$ and define $\beta = \widehat{D}\phi$, so we have

$$\int_{\widetilde{M}} \langle \beta, \widehat{D}\psi \rangle = 0 \text{ for all } \psi \in W^{1,2}.$$

This implies $\widehat{D}\beta = -n\sqrt{-1}\beta$ weakly, as $\widehat{D}^* = \widehat{D} + n\sqrt{-1}$.

As argued in [34] Lemma 3.3, $\beta \in W^{1,2}_{loc}$. Note also that in the weak sense, $\widehat{D}\beta = -n\sqrt{-1}\beta = -n\sqrt{-1}(\widehat{D}\phi_1 - \widehat{D}\phi_0) \in L^2$. Then

$$\begin{aligned}
\int_{\widetilde{M}_\rho} \langle \widehat{D}\beta, \widehat{D}\beta \rangle &= \int_{\widetilde{M}_\rho} \langle (\widehat{D} + n\sqrt{-1})\widehat{D}\beta, D\beta \rangle - \int_{\Sigma_\rho} \langle \widetilde{c}(\nu)\widehat{D}\beta, \widehat{D}\beta \rangle \\
&= \int_{\widetilde{M}_\rho} \langle \widehat{D}(\widehat{D} + n\sqrt{-1})\beta, \beta \rangle - \int_{\Sigma_\rho} \langle \widetilde{c}(\nu)\widehat{D}\beta, \widehat{D}\beta \rangle \\
&= -\int_{\Sigma_\rho} \langle \widetilde{c}(\nu)\widehat{D}\beta, \widehat{D}\beta \rangle \\
&\leq \int_{\Sigma_\rho} |\widehat{D}\beta|^2.
\end{aligned}$$

As $\displaystyle\int_{\widetilde{M}} |\widehat{D}\beta|^2 < \infty$, there is a sequence $\rho_m \to \infty$ such that $\int_{\Sigma_{\rho_m}} |\widehat{D}\beta|^2 \to 0$. But then

$$\int_{\widetilde{M}} |\widehat{D}\beta|^2 = \lim_{m\to\infty} \int_{\widetilde{M}_{\rho_m}} |\widehat{D}\beta|^2 \leq \lim_{m\to\infty} \int_{\Sigma_{\rho_m}} |\widehat{D}\beta|^2 \to 0.$$

i.e. $\widehat{D}\beta = 0$. As $\widehat{D}\beta = -n\sqrt{-1}\beta$, we have

$$\widehat{D}\phi = \beta = 0.$$

Now, by the Lichnerowicz formula (3.17), as $\widehat{D}\phi = 0$,

$$\begin{aligned}
0 &\leq \int_{\widetilde{M}_\rho} \left(|\widehat{\nabla}\phi|^2 + \frac{1}{4}(R + n(n-1))|\phi|^2 \right) \\
&= \int_{\Sigma_\rho} \langle \widehat{B}\phi, \phi \rangle \\
&= \int_{\Sigma_\rho} \langle \widehat{B}(\phi_1 - \phi_0), \phi_1 - \phi_0 \rangle \\
&= \int_{\Sigma_\rho} \langle \widehat{B}\phi_0, \phi_0 \rangle + \left(\int_{\Sigma_\rho} \langle \widehat{B}\phi_1, \phi_1 \rangle - \int_{\Sigma_\rho} \langle \widehat{B}\phi_0, \phi_1 \rangle - \int_{\Sigma_\rho} \langle \widehat{B}\phi_1, \phi_0 \rangle \right).
\end{aligned}$$

We claim that there is $\rho_m \to \infty$ such that the three terms in the bracket above will tend to zero as $m \to \infty$. Consider

$$\begin{aligned}\int_{\Sigma_\rho} \langle \widehat{B}\phi_0, \phi_1\rangle &= \int_{\Sigma_\rho} \langle (\widehat{\nabla}_\nu + \widetilde{c}(\nu)\widehat{D})\phi_0, \phi_1\rangle \\ &\le (\int_{\Sigma_\rho} |\widehat{\nabla}_\nu \phi_0|^2)^{\frac{1}{2}}(\int_{\Sigma_\rho} |\phi_1|^2)^{\frac{1}{2}} + (\int_{\Sigma_\rho} |\widehat{D}\phi_0|^2)^{\frac{1}{2}}(\int_{\Sigma_\rho} |\phi_1|^2)^{\frac{1}{2}}.\end{aligned}$$

As $\int_{\widetilde{M}} |\widehat{\nabla}\phi_0|^2$, $\int_{\widetilde{M}} |\widehat{D}\phi_0|^2$ and $\int_{\widetilde{M}} |\phi_1|^2$ are all finite, there is $\rho_m \to \infty$ such that $\int_{\Sigma_{\rho_m}} \langle \widehat{B}\phi_0, \phi_1\rangle \to 0$. Similarly, as $\int_{\widetilde{M}} |\widehat{\phi}_1|^2 < \infty$, we can also assume that $\int_{\Sigma_{\rho_m}} \langle \widehat{B}\phi_1, \phi_1\rangle \to 0$. (3.21) is proved.

Now, by Proposition 3.13 and 3.14,

$$0 \le \lim_{m\to\infty} \int_{\Sigma_{\rho_m}} \langle \widehat{B}\phi, \phi\rangle = \lim_{m\to\infty} \int_{\Sigma_{\rho_m}} \langle \widehat{B}\phi_0, \phi_0\rangle = \lim_{m\to\infty} \frac{1}{2}\int_{\Sigma_{\rho_m}} (H_0 - H)|\phi_0|^2_{\widetilde{g}}.$$

By Proposition 3.15, $\lim_{\rho\to\infty} \frac{1}{2}\int_{\Sigma_\rho} (H_0 - H)|\phi_0|^2_{\widetilde{g}}$ exists, therefore

$$\lim_{\rho\to\infty} \frac{1}{2}\int_{\Sigma_\rho} (H_0 - H)|\phi_0|^2_{\widetilde{g}} \ge 0.$$

To finish the proof, by Proposition 3.10, we can let $\zeta = \zeta_a$. Let $\phi'_{a,0}$ be the corresponding Killing spinor on $\mathbb{H}^n_{-k^2}$ and $\phi_{a,0} = A\phi'_{a,0}$ outside Ω. By Proposition 3.9, $|\phi_{a,0}|^2_{\widetilde{g}} = |\phi'_{a,0}|^2_{g'} = -2kX \cdot \zeta_a$. So the above argument shows that

$$-k \lim_{\rho\to\infty} \int_{\Sigma_\rho} (H_0 - H)X \cdot \zeta_a \ge 0.$$

In other words, $\lim_{\rho\to\infty} \int_{\Sigma_\rho} (H_0 - H)X d\Sigma_\rho$ is a future-directed non-spacelike vector.

□

3.3 Positivity of Shi-Tam mass

Now assume $n \ge 3$ and let (Ω, g) be as described in section 3.1. Recall that $B_0(R_1)$ and $B_0(R_2)$ are geodesic balls in $\mathbb{H}^n_{-k^2}$ such that $B_0(R_1) \subset \Omega_0 \subset B_0(R_2)$.

Theorem 3.16. *(cf. [35] Theorem 3.1) Let $n \geq 3$ and (Ω, g) be a compact spin n-manifold with smooth boundary Σ. Assuming the following conditions:*

1. *The scalar curvature R of (Ω, g) satisfies $R > -n(n-1)k^2$ for some $k > 0$,*
2. *Σ is topologically a $(n-1)$-sphere with sectional curvature $K > -k^2$, mean curvature $H > 0$ and Σ can be isometrically embedded uniquely into $\mathbb{H}^n_{-k^2}$ with mean curvature H_0.*

Then for any future directed null vector ζ in $\mathbb{R}^{n,1}$,

$$m(\Omega, \zeta) = \int_\Sigma (H_0 - H) W \cdot \zeta \leq 0$$

where $W = (x_1, x_2, \cdots, x_n, \alpha t)$ with

$$1 < \alpha = \coth kR_1 + \frac{1}{\sinh kR_1}\left(\frac{\sinh^2 kR_2}{\sinh^2 kR_1} - 1\right)^{\frac{1}{2}},$$

$X = (x_1, x_2, \cdots, x_n, t)$ is the position vector in $\mathbb{R}^{n,1}$ and the inner product is given by the Lorentz metric.

Let $(\phi_1, \cdots, \phi_n)$ denote the position vectors of points of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Let $\{\Sigma_\rho\}$ be the foliation of $\mathbb{H}^n_{-k^2} \setminus \Omega_0$ described in section 3.1. We need the following:

Lemma 3.17. *With the assumptions in Theorem 3.16, let $(y_1, \cdots, y_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n y_i^2 = 1$. Let $\phi = \sum_{i=1}^n \phi_i y_i$. Then for $\rho > 0$,*

$$(\frac{\partial \phi}{\partial \rho})^2 \leq (1 - \phi^2) k^2 \sinh^{-2} kr \left(1 - (\frac{\partial r}{\partial \rho})^2\right).$$

Hence

$$\left|\frac{\partial \phi}{\partial \rho}\right| \leq \mu k \left|\frac{\partial r}{\partial \rho}\right| \tag{3.22}$$

where

$$\mu = \frac{1}{\sinh kR_1}\left(\frac{\sinh^2 kR_2}{\sinh^2 kR_1} - 1\right)^{\frac{1}{2}}.$$

Proof. The same as in [35] Lemma 3.3 (ii). The position vectors in $\mathbb{R}^{n,1}$ can be parametrized by

$$X = \frac{1}{k}(\sinh kr\cos\theta, \sinh kr\sin\theta\vec{z}, \cosh kr)$$

where $\vec{z} \in \mathbb{S}^{n-2} \subset \mathbb{R}^{n-1}$. Then the hyperbolic metric (outside Ω_0) is

$$g' = d\rho^2 + g_\rho = dr^2 + k^{-2}\sinh^2 r(d\theta^2 + \sin^2\theta d\sigma)$$

where $d\sigma$ is the standard metric on $\mathbb{S}^{n-2}$. Compute $g'(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho})$ in the above two forms of g', we have

$$1 = (\frac{\partial r}{\partial\rho})^2 + k^{-2}\sinh^2\left[(\frac{\partial\theta}{\partial\rho})^2 + \sin^2\theta d\sigma(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\rho})\right] \geq (\frac{\partial r}{\partial\rho})^2 + k^{-2}\sinh^2(\frac{\partial\theta}{\partial\rho})^2.$$

Since $\phi = \cos\theta$, the conclusion follows. □

Proof of Theorem 3.16. X can be expressed as

$$X = \frac{1}{k}(\sinh(kr)Y, \cosh kr) = \frac{1}{k}(\sinh(kr)y_1, \cdots, \sinh(kr)y_n, \cosh kr).$$

where $|Y|^2 = \sum_{i=1}^n y_i^2 = 1$. Without loss of generality we can assume that $\zeta = (\zeta_1, \cdots, \zeta_n, 1)$ where $\sum_{i=1}^n \zeta_i^2 = 1$.

Let $\phi = \sum_{i=1}^n y_i\zeta_i$, then Lemma 3.6 implies (we omit $d\Sigma_\rho$ for convenience)

$$\begin{aligned}
&\frac{d}{d\rho}(\int_{\Sigma_\rho}(H_0 - H)W(\rho, p)\cdot\zeta) \\
=& -\int_{\Sigma_\rho} u^{-1}(u-1)^2(\frac{1}{2}(R^\rho + (n-1)(n-2)k^2)(\phi\sinh kr - \alpha\cosh kr) \\
&+ H_0\frac{\partial}{\partial\rho}(\phi\sinh kr - \alpha\cosh kr)) \\
=& -\int_{\Sigma_\rho} u^{-1}(u-1)^2(\frac{1}{2}(H_0^2 - |A|^2)(\phi\sinh kr - \alpha\cosh kr) \\
&+ H_0\frac{\partial}{\partial\rho}(\phi\sinh kr - \alpha\cosh kr)) \\
=& -\int_{\Sigma_\rho} u^{-1}(u-1)^2 B \quad \text{where}
\end{aligned} \tag{3.23}$$

$$
\begin{aligned}
B =& \frac{1}{2}(H_0^2 - |A|^2)(\phi \sinh kr - \alpha \cosh kr) \\
&+ kH_0(\phi \cosh kr \frac{\partial r}{\partial \rho} + \frac{1}{k} \sinh kr \frac{\partial \phi}{\partial \rho} - \alpha \sinh kr \frac{\partial r}{\partial \rho}).
\end{aligned} \tag{3.24}
$$

Here A is the second fundamental form of Σ_ρ with respect to the hyperbolic metric. Let $\lambda_a(p, \rho)$ be the principal curvature of Σ_ρ. Then $\lambda_a = k \tanh k(\mu_a + \rho)$, k, or $k \coth k(\mu_a + \rho)$ with $\mu_a > 0$ ([38] p.255). In particular,

$$H_0^2 - |A|^2 = 2 \sum_{a<b} \lambda_a \lambda_b \geq 0.$$

We want to show $B \leq 0$. For the first term of B, consider

$$\phi \sinh kr - \alpha \cosh kr \leq \sinh kr - \cosh kr < 0. \tag{3.25}$$

To show that the last term of R.H.S. of (3.24) is also negative, it suffices to show

$$\phi \cosh kr \frac{\partial r}{\partial \rho} + \frac{1}{k} \sinh kr \frac{\partial \phi}{\partial \rho} - \alpha \sinh kr \frac{\partial r}{\partial \rho} < 0.$$

Recall that $o \in \Omega_0$ and r is the geodesic distance from o. Let $p \in \Sigma$ and let γ be the (arc-length parametrized) geodesic through p which is orthogonal to Σ. Let q be the point on γ such that $a = d(o, q) = d(o, \gamma)$. Since the last term of R.H.S. of (3.24) involves only the derivatives with respect to ρ, we can assume that $\gamma(0) = q$ and $\gamma(\rho_0) = p$ for some positive ρ_0, so that ρ is the geodesic distance from q to $\gamma(\rho)$. Denote the geodesic from x to y to be xy.

If $o \neq q$, then oq and qp forms a right angle at q. That is, o, q and $\gamma(\rho)$ forms a right-angled triangle on the totally geodesic $\mathbb{H}^2_{-k^2}$ containing them with sides a, ρ and hypotenuse r.

The cosine law $\cosh kr = \cosh ka \cosh k\rho$ implies

$$\frac{\partial r}{\partial \rho} = \frac{\cosh ka \sinh k\rho}{\sinh kr} > 0.$$

If $o = q$, then $r = \rho$ and clearly $\frac{\partial r}{\partial \rho} = 1 > 0$. So by (3.22),

$$\begin{aligned}
&\phi \cosh kr\frac{\partial r}{\partial \rho} + \frac{1}{k}\sinh kr\frac{\partial \phi}{\partial \rho} - \alpha \sinh kr\frac{\partial r}{\partial \rho} \\
\leq &\cosh kr\frac{\partial r}{\partial \rho} + \frac{1}{k}\sinh kr(\mu k\frac{\partial r}{\partial \rho}) - \alpha \sinh kr\frac{\partial r}{\partial \rho} \\
= &(\cosh kr + \sinh kr(\mu - \alpha))\frac{\partial r}{\partial \rho} \\
= &(\cosh kr - \sinh kr \coth kR_1)\frac{\partial r}{\partial \rho} \\
< &0. \quad (\text{as } r > R_1)
\end{aligned}$$

Substituting into (3.23), we have

$$\frac{d}{d\rho}(\int_{\Sigma_\rho}(H_0 - H)W(\rho, p) \cdot \zeta) \geq 0.$$

By Theorem 3.7 and Corollary 3.8, we conclude that

$$m(\Omega, \zeta) = \int_{\Sigma}(H_0 - H)W \cdot \zeta \leq 0.$$

$\square$

Bibliography

[1] L. Andersson and M. Dahl. Scalar curvature rigidity for asymptotically locally hyperbolic manifolds. *Ann. Global Anal. Geom.*, 16(1):1–27, 1998.

[2] R. Arnowitt, S. Deser, and C. Misner. Coordinate invariance and energy expressions in general relativity. *Physical Review*, 122(3):997, 1961.

[3] R. Bartnik. The mass of an asymptotically flat manifold. *Comm. Pure Appl. Math.*, 39(5):661–693, 1986.

[4] D. Baskaran, S. Lau, and A. Petrov. Center of mass integral in canonical general relativity. *Annals of Physics*, 307(1):90–131, 2003.

[5] H. Baum. Complete Riemannian manifolds with imaginary Killing spinors. *Ann. Global Anal. Geom.*, 7(3):205–226, 1989.

[6] H. Braden, J. Brown, B. Whiting, and J. York Jr. Charged black hole in a grand canonical ensemble. *Physical Review D*, 42(10):3376, 1990.

[7] J. Brown and J. York Jr. Quasilocal energy in general relativity. In *Mathematical aspects of classical field theory: proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held July 20-26, 1991, with support from the National Science Foundation*, volume 132, page 129. Amer Mathematical Society, 1992.

[8] J. Brown and J. York Jr. Quasilocal energy and conserved charges derived from the gravitational action. *Physical Review D*, 47(4):1407, 1993.

[9] P. Chrusciel and M. Herzlich. The mass of asymptotically hyperbolic Riemannian manifolds. *Pacific Journal of Mathematics*, 212(2), 2003.

[10] M. Do Carmo. *Differential geometry of curves and surfaces*, volume 2. Prentice-Hall Englewood Cliffs, NJ:, 1976.

[11] X.-Q. Fan and K.-K. Kwong. The Brown-York mass of revolution surfaces in asymptotically Schwarzschild manifolds. *J. Geom. Anal.*, 21(3):527–542, 2011.

[12] X.-Q. Fan, Y.-G. Shi, and L.-F. Tam. Large-sphere and small-sphere limits of the Brown-York mass. *Comm. Anal. Geom.*, 17(1):37–72, 2009.

[13] S. Hawking. Gravitational radiation in an expanding universe. *J. Math. Phys.*, 9:598, 1968.

[14] S. Hawking and G. Horowitz. The Gravitational Hamiltonian, action, entropy and surface terms. *Classical and Quantum Gravity*, 13:1487, 1996.

[15] G. Herglotz. Über die steinersche formel für parallelflächen. In *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, volume 15, pages 165–177. Springer, 1943.

[16] O. Hijazi, S. Montiel, and A. Roldán. Dirac Operators on Hypersurfaces of Manifolds with Negative Scalar Curvature. *Ann. Global Anal. Geom.*, 23(3):247–264, 2003.

[17] R. Howard. Blaschke's rolling theorem for manifolds with boundary. *Manuscripta Mathematica*, 99(4):471–483, 1999.

[18] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *J. Differential Geom.*, 59(3):353–437, 2001.

[19] G. Huisken and S.-T. Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Inventiones Mathematicae*, 124(1):281–311, 1996.

[20] K.-K. Kwong and L.-F. Tam. Limit of quasilocal mass integrals in asymptotically hyperbolic manifolds. *Arxiv preprint arXiv:1010.6130*, 2010, to appear in Proc. of the AMS.

[21] H. Lawson and M. Michelsohn. *Spin geometry*. Princeton Univ Pr, 1989.

[22] Y. Li and G. Weinstein. A priori bounds for co-dimension one isometric embeddings. *American Journal of Mathematics*, 121(5):945–965, 1999.

[23] C.-C. M. Liu and S.-T. Yau. Positivity of quasilocal mass. *Physical Review Letters*, 90(23):231102, 2003.

[24] P. Miao, Y.-G. Shi, and L.-F. Tam. On geometric problems related to Brown-York and Liu-Yau quasilocal mass. *Comm. Math. Phys.*, pages 1–23.

[25] L. Nirenberg. The Weyl and Minkowski problems in differential geometry in the large. *Comm. Pure Appl. Math.*, 6(3):337–394, 1953.

[26] T. Parker and C. Taubes. On Witten's proof of the positive energy theorem. *Comm. Math. Phys.*, 84(2):223–238, 1982.

[27] R. Penrose. Quasi-local mass and angular momentum in general relativity. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, 381(1780):53, 1982.

[28] A. Pogorelov. Some results on surface theory in the large. *Advances in Math.*, 1(2):191–264, 1964.

[29] A. Pogorelov. *Extrinsic geometry of convex surfaces*. Amer. Mathematical Society, 1973.

[30] R. Sacksteder. The rigidity of hypersurfaces. *J. Math. Mech*, 11(27):929–939, 1962.

[31] R. Schoen and S.-T. Yau. On the proof of the positive mass conjecture in general relativity. *Comm. Math. Phys.*, 65(1):45–76, 1979.

[32] R. Schoen and S.-T. Yau. Proof of the positive mass theorem. II. *Comm. Math. Phys.*, 79(2):231–260, 1981.

[33] R. Schoen and S.-T. Yau. The energy and the linear momentum of space-times in general relativity. *Comm. Math. Phys.*, 79(1):47–51, 1981.

[34] Y.-G. Shi and L.-F. Tam. Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature. *J. Differential Geom.*, 62(1):79–125, 2002.

[35] Y.-G. Shi and L.-F. Tam. Rigidity of compact manifolds and positivity of quasi-local mass. *Classical and Quantum Gravity*, 24:2357, 2007.

[36] Y.-G. Shi, G.-F. Wang, and J. Wu. On the behavior of quasi-local mass at the infinity along nearly round surfaces. *Ann. Global Anal. Geom.*, 36(4):419–441, 2009.

[37] A. Treibergs. Private communications.

[38] M.-T. Wang and S.-T. Yau. A generalization of Liu-Yau's quasi-local mass. *Comm. Anal. Geom.*, 15(2):249–282, 2007.

[39] M.-T. Wang and S.-T. Yau. Quasilocal mass in general relativity. *Physical review letters*, 102(2):21101, 2009.

[40] X.-D. Wang. The mass of asymptotically hyperbolic manifolds. *J. Differential Geom.*, 57(2):273–299, 2001.

[41] E. Witten. A new proof of the positive energy theorem. *Comm. Math. Phys.*, 80(3):381–402, 1981.

[42] X. Zhang. A definition of total energy-momenta and the positive mass theorem on asymptotically hyperbolic 3-manifolds. I. *Comm. Math. Phys.*, 249(3):529–548, 2004.

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